# NON-CLASSICAL ORTHOGONAL POLYNOMIALS

By
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# DEDICATION

This Dissertation is affectionately
dedicated to the son of the writer
George Newell Young

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# PREFACE

The purpose of this dissertation is to study the Non-Classical Orthogonal Polynomial solutions of that class of second order linear differential equations with parameter which have a recursion relation involving two coefficients of the solution by the method of Frobenius.

It is convenient to take the differential equation in the form

$$(A_1x^{r+2} + A_2x^{s+2})y''_n + (B_1x^{r+1} + B_2x^{s+1})y'_n + \left[L_1(n)x^r + L_2(n)x^s\right]y_n = 0$$

where $A_1$, $A_2$, $B_1$, $B_2$ are rational constants, and $L_1(n)$ and $L_2(n)$ may be functions of the integral parameter $n$.

Conditions are obtained for the existence of the two types of such solutions which arise, together with their appropriate weight functions and intervals of orthogonality, formulas are developed for obtaining these solutions and their integrated squares or normalizing constants under certain circumstances, and some of their interesting properties are considered. Several numerical examples illustrate the use of these results. The relation of the study to the boundary value problems of mathematical physics is indicated.

The work divides itself naturally into four parts, in each of which important results are numbered in sequence for reference, successive primes indicating successive repetitions of a given result. References to bibliography are by number.

# ABSTRACT

of

# NON-CLASSICAL ORTHOGONAL POLYNOMIALS

by

John W. Young

Investigation is made of the conditions under which the class of second order homogeneous linear differential equations

$$(A_1x^{r+2} + A_2x^{s+2})y''_n + (B_1x^{r+1} + B_2x^{s+1})y'_n + \left[L_1(n)x^r + L_2(n)x^s\right]y_n = 0$$

possesses orthogonal polynomial solutions. It is shown that the classical orthogonal polynomials, such as those of Legendre, Hermite, and Laguerre, are contained as special cases.

Particular attention is given to the non-classical orthogonal polynomial solutions obtained when $L_1(n) = K_1 L(n)$, $L_2(n) = K_2 L(n)$. It is found that there are then two general types of such solutions, one of order and subdegree linear in n, and having an algebraic weight function and finite interval of orthogonality; the other of order quadratic in n, and of subdegree linear in n, and having a transcendental weight function and semi-infinite interval of orthogonality. With suitable restrictions, derivative definitions and integrated square formulas are obtained for these solutions. Certain other properties of interest in these solutions are also considered.

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# PART I

# NATURE OF THE PROBLEM
and
DEVELOPMENT OF PRELIMINARY RESULTS

## SECTION 1

## ORTHOGONALITY OF THE SOLUTIONS WITH INTEGRATING FACTOR

The Non-Classical Orthogonal Polynomials which are the subject of this dissertation are those which under conditions to be studied arise as solutions of the second order homogeneous linear differential equation with rational constant coefficients

$$(A_1 x^{r+2} + A_2 x^{s+2})y''_n + (B_1 x^{r+1} + B_2 x^{s+1})y'_n + \left[L_1(n)x^r + L_2(n)x^s\right]y_n = 0 \qquad \text{I-1}$$

where $L_1(n)$ and $L_2(n)$ may be functions of the integral parameter $n \geq 0$. With no loss of generality we may take these coefficients to be integers, with $A_1 > 0$.

To simplify the handling of this equation we let

$$p(x) = A_1 x^{r+2} + A_2 x^{s+2} \qquad \text{I-2}$$

$$q(x) = B_1 x^{r+1} + B_2 x^{s+1} \qquad \text{I-3}$$

$$t_n(x) = L_1(n)x^r + L_2(n)x^s \qquad \text{I-4}$$

We also let

$$w(x) = \text{an integrating factor} \qquad \text{I-5}$$

The differential equation I-1 then becomes

$$py''_n + qy'_n + t_n y_n = 0 \qquad \text{I-6}$$

Multiplying two such equations by w and each equation by the solution of the other, we have by addition

$$\begin{array}{r|l} -y_m & wpy''_n + wqy'_n + wt_n y_n = 0 \\ y_n & wpy''_m + wqy'_n + wt_m y_m = 0 \\ \hline \end{array}$$

$$wp(y''_m y_n - y_m y''_n) + wq(y'_m y_n - y_m y'_n) + w(t_m - t_n)y_m y_n = 0$$

But we observe that

$$\frac{d}{dx}\left[wp(y'_m y_n - y_m y'_n)\right] = wp(y''_m y_n + y'_m y'_n - y'_m y'_n - y_m y''_n) + \frac{d}{dx}(wp)(y'_m y_n - y_m y'_n)$$

Accordingly, integration of our preceding result from $x_0$ to $x_1$ gives

$$\left[wp(y'_m y_n - y_m y'_n)\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} w(t_m - t_n)y_m y_n \, dx = 0 \tag{I-7}$$

provided that

$$\frac{d}{dx}(wp) = wq \tag{I-8}$$

Dividing by wp and multiplying by dx, we have

$$\frac{d(wp)}{wp} = \frac{q}{p}\,dx$$

By integration we have

$$\ln (wp) = \int \frac{q}{p}\, dx$$

Changing to exponential notation gives

$$wp = e^{\int \frac{q}{p}\, dx} \qquad \text{I-9}$$

Hence the integrating factor is

$$w = \frac{1}{p}\, e^{\int \frac{q}{p}\, dx} \qquad \text{I-10}$$

If this integrating factor $w = 1$, so that from I-8

$$\frac{dp}{dx} = q$$

then the differential equation will be called self-adjoint.

Now from I-7 we see that we will have

$$\int_{x_0}^{x_1} w(t_m - t_n) y_m y_n \, dx = 0, \; m \neq n \qquad \text{I-11}$$

where $x_0$ and $x_1$ are roots of (wp), if y and y' are finite at $x_0$ and $x_1$, and if wp, y, y', y", t are continuous in $(x_0, x_1)$

so that the integrals exist. These requirements are surely met if $y_n$ are polynomials, and $x_o$, $x_1$ are finite.

The property of $y_n$ expressed by I-11 will be called the orthogonality property, and the integral I-11 itself the orthogonality relation.

## SECTION 2

## NON-VANISHING OF THE INTEGRATED SQUARE FOR $n \neq 0$

Making use of the relation $wq = (wp)'$ from I-8, and multiplying the differential equation I-6 by $wy_n$, we have

$$(wp)y_n''y_n + (wp)'y_n'y_n + wt_ny_n^2 = 0$$

But we observe that

$$\frac{d}{dx}\left[(wp)y_n'y_n\right] = (wp)'y_n'y_n + (wp)y_n''y_n + (wp){y_n'}^2$$

Accordingly, integration of our previous result from $x_o$ to $x_1$ gives

$$\left[(wp)y_n'y_n\right]_{x_o}^{x_1} - \int_{x_o}^{x_1} (wp){y_n'}^2\,dx + \int_{x_o}^{x_1} wt_ny_n^2\,dx = 0$$

The bracket vanishes under the same conditions as those required for the vanishing of the bracket in I-7 so as to establish the orthogonality relation I-11. Hence we have

$$\int_{x_0}^{x_1} (wp){y'_n}^2 \, dx = \int_{x_0}^{x_1} wt_n y_n^2 \, dx \qquad \text{I-12}$$

Now if in addition to the preceding conditions there are no roots of wp between $x_0$ and $x_1$, so that wp is of non-changing sign between $x_0$ and $x_1$, then the integral on the left is $\neq 0$, unless $y_n$ = constant, so that we would have $y'_n = 0$. Hence the integral on the right is $\neq 0$, whether or not $t_n$ has roots between $x_0$ and $x_1$, provided $n \neq 0$. As will be shown later in II-97 and II-97'(ff.III-74), we have $y_0$ = constant, so that $y'_0 = 0$. Since for $n = 0$ it also turns out that $t_0 = 0 \cdot \phi(x)$, $\phi$ a function of x, then I-12 becomes

$$0 = 0 \cdot \int_{x_0}^{x_1} w \, \phi(x) \, y_0^2 \, dx \qquad \text{I-13}$$

so that the integral itself may or may not be zero. Hence we must establish further conditions for the non-vanishing of the integrated square when $n = 0$. This is done in Part I, Sections 14 and 17; Part II, Section 9; and Part III, Section 9.

## SECTION 3

## EXPANSION OF AN ARBITRARY FUNCTION IN SERIES OF THE SOLUTIONS WITH WEIGHT FUNCTION

Designate the arbitrary function by $f(x)$ and assume

$$f(x) = \sum_{n=0}^{\infty} C_n y_n \qquad \text{I-14}$$

Multiplying by $w(t_n - t_i)y_m$ , and integrating between $x_0$ and $x_1$, we have

$$\int_{x_0}^{x_1} w(t_n - t_i)y_m f(x)\,dx = \sum_{n=0}^{\infty} C_n \int_{x_0}^{x_1} w(t_n - t_i)y_m y_n\,dx$$

By I-7 we can say only that the integrals on the right are necessarily zero for $n = i$. Hence the right member does not necessarily reduce to a single integral as is required in order to find $C_n$. Thus it is clear that what is required is an orthogonality relation which does not involve the parameter n of $t_n$. To obtain this we use I-4 in I-11 and get

$$\int_{x_0}^{x_1} w\left\{\left[L_1(m)x^r + L_2(m)x^s\right] - \left[L_1(n)x^r + L_2(n)x^s\right]\right\} y_m y_n\,dx = 0$$

Rearranging this we have

$$\left[L_1(m) - L_1(n)\right]\int_{x_0}^{x_1} wx^r y_m y_n \, dx = \left[L_2(n) - L_2(m)\right]\int_{x_0}^{x_1} wx^s y_m y_n \, dx \qquad \text{I-15}$$

Three cases of an orthogonality relation independent of the n in $t_n$ now arise. We consider each of these separately:

Case I: $L_1(n) = L_1$, a constant independent of n. I-16

The left member of I-15 is then zero. Hence for $L_2(m) \neq L_2(n)$ we have the orthogonality relation

$$\int_{x_0}^{x_1} w\, x^s \, y_m y_n \, dx = 0, \; m \neq n \qquad \text{I-17}$$

independent of the parameter n in $t_n$, the weight function W, the function with respect to which the orthogonality relation exists, being

$$W = w\, x^s \qquad \text{I-18}$$

Case II: $L_2(n) = L_2$, a constant independent of n. I-19

The right member of I-15 is then zero. Hence, for $L_1(m) \neq L_1(n)$ we have the orthogonality relation

$$\int_{x_0}^{x_1} w\, x^r\, y_m y_n\, dx = 0, \; m \neq n \qquad \text{I-20}$$

independent of the parameter n in $t_n$, with weight function

$$W = w\, x^r \qquad \text{I-21}$$

Case III: $L_1(n) = K_1\, L(n),\; L_2(n) = K_2\, L(n)$ I-22

where the K's are constants independent of n.

Using this in I-4 and the result in I-11, we have

$$\left[L(m) - L(n)\right] \int_{x_0}^{x_1} w(K_1 x^r + K_2 x^s)\, y_m y_n\, dx = 0$$

so that for $L(m) \neq L(n)$ we have the orthogonality relation

$$\int_{x_0}^{x_1} w(K_1 x^r + K_2 x^s)\, y_m y_n\, dx = 0, \quad m \neq n \qquad \text{I-23}$$

independent of the parameter n in $t_n$, with weight function

$$W = w(K_1 x^r + K_2 x^s) \qquad \text{I-24}$$

Accordingly, combining the results of I-17, I-20, I-23, we have the orthogonality relation

$$\int_{x_0}^{x_1} W\, y_m y_n \, dx = 0, \quad m \neq n \qquad \text{I-25}$$

where W is taken as in I-18, I-21, I-24, according as I-16, I-19, I-22 holds.

## SECTION 4

## RECURSION RELATION AND RECURSION PRODUCT FOR THE DESCENDING SERIES SOLUTIONS

Using the method of Frobenius, we assume the solution of the form

$$y_n = \sum_{\theta=0}^{\infty} c^{n}_{k+\theta j}\, x^{k+\theta j} \qquad \text{I-26}$$

Writing the differential equation I-1 in tabular form after dividing by $x^s$, and taking $s > r$, letting the jump

$$j = r - s < 0 \qquad \text{I-27}$$

we have

$$\begin{array}{llll|l} L_1(n)x^{j} & & + L_2(n) & & y_n \\ B_1 & x^{j+1} & + B_2 & x & y'_n \\ A_1 & x^{j+2} & + A_2 & x^2 & y''_n \end{array} \qquad \text{I-28}$$

From I-26 we have by differentiation

$$y_n = \dots + c^n_{k+\theta j}\, x^{k+\theta j} + c^n_{k+(\theta+1)j}\, x^{k+(\theta+1)j} + \dots$$

$$y'_n = \dots + (k+\theta j)\, c^n_{k+\theta j}\, x^{k+\theta j-1}$$
$$+ \left[k+(\theta+1)j\right] c^n_{k+(\theta+1)j}\, x^{k+(\theta+1)j-1} + \dots$$

$$y''_n = \dots + (k+\theta j)(k+\theta j-1)\, c^n_{k+\theta j}\, x^{k+\theta j-2}$$
$$+ \left[k+(\theta+1)j\right]\left[k+(\theta+1)j-1\right] c^n_{k+(\theta+1)j}\, x^{k+(\theta+1)j-2} + \dots \qquad \text{I-29}$$

Substituting these results in I-28 we have

$$\dots + x^{k+(\theta+1)j}\left[c^n_{k+\theta j}\left\{L_1(n) + B_1(k+\theta j) + A_1(k+\theta j)(k+\theta j-1)\right\}\right.$$
$$\left. + c^n_{k+(\theta+1)j}\left\{L_2(n) + B_2\left[k+(\theta+1)j\right] + A_2\left[k+(\theta+1)j\right]\left[k+(\theta+1)j-1\right]\right\}\right] + \dots \overset{(x)}{\equiv} 0$$

Hence the recursion relation for the descending series solutions

$$c^n_{k+(\theta+1)j} = - \frac{A_1(k+\theta j)(k+\theta j-1) + B_1(k+\theta j) + L_1(n)}{A_2\left[k+(\theta+1)j\right]\left[k+(\theta+1)j-1\right] + B_2\left[k+(\theta+1)j\right] + L_2(n)}\, c^n_{k+\theta j}$$

$$= - \frac{A_1(k+\theta j)^2 + (B_1-A_1)(k+\theta j) + L_1(n)}{A_2\left[k+(\theta+1)j\right]^2 + (B_2-A_2)\left[k+(\theta+1)j\right] + L_2(n)}\, c^n_{k+\theta j}$$

From this we have the recursion product for the descending series solutions

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_1(k+\phi j)^2 + (B_1-A_1)(k+\phi j) + L_1(n)}{\prod_{\phi=1}^{\theta} A_2(k+\phi j)^2 + (B_2-A_2)(k+\phi j) + L_2(n)}\, c^n_k \qquad \text{I-30}$$

wherein we define

$$\prod_{0}^{-1} = \prod_{1}^{0} = 1$$

## SECTION 5

## RECURSION RELATION AND RECURSION PRODUCT FOR THE ASCENDING SERIES SOLUTIONS

Again using the method of Froberius, we assume the solution to be of the form I-26. Writing the differential equation I-1 in tabular form after dividing by $x^r$, and taking $s > r$ as before, but this time letting

$$j = s-r > 0 \qquad \text{I-31}$$

we have

$$\begin{array}{llll|l} L_1(n) & & + L_2(n) & x^j & y_n \\ B_1 & x & + B_2 & x^{j+1} & y'_n \\ A_1 & x^2 & + A_2 & x^{j+2} & y''_n \end{array} \qquad \text{I-32}$$

Substituting the results I-29 in this we have

$$\dots + x^{k+(\theta+1)j}\left( c^n_{k+\theta j}\left\{L_2(n) + B_2(k+\theta j) + A_2(k+\theta j)(k+\theta j-1)\right\}\right.$$
$$\left. + c^n_{k+(\theta+1)j}\left\{L_1(n) + B_1\left[k+(\theta+1)j\right] + A_1\left[k+(\theta+1)j\right]\left[k+(\theta+1)j-1\right]\right\}\right) + \dots \overset{(x)}{=} 0$$

Hence the recursion relation for the ascending series solutions

$$c^n_{k+(\theta+1)j} = -\frac{A_2(k+\theta j)(k+\theta j-1) + B_2(k+\theta j) + L_2(n)}{A_1\left[k+(\theta+1)j\right]\left[k+(\theta+1)j-1\right] + B_1\left[k+(\theta+1)j\right] + L_1(n)}\, c^n_{k+\theta j}$$

$$= - \frac{A_2(k+\theta j)^2 + (B_2 - A_2)(k+\theta j) + L_2(n)}{A_1\left[k+(\theta+1)j\right]^2 + (B_1 - A_1)\left[k+(\theta+1)j\right] + L_1(n)} \, c^n_{k+\theta j}$$

From this we have the recursion product for the ascending series solutions

$$c^n_{k+\theta j} = (-1)^\theta \, \frac{\prod_{\phi=0}^{\theta-1} A_2(k+\phi j)^2 + (B_2 - A_2)(k+\phi j) + L_2(n)}{\prod_{\phi=1}^{\theta} A_1(k+\phi j)^2 + (B_1 - A_1)(k+\phi j) + L_1(n)} \, c^n_k \qquad \text{I-33}$$

wherein, as following I-30, we again define

$$\prod_{0}^{-1} = \prod_{1}^{0} = 1$$

We observe that with this notation the recursion product for the ascending and descending series solution has the same form, except that the subscripts are interchanged. Accordingly, results obtained for the descending series can be immediately applied to the ascending series merely by interchanging the subscripts in the results and taking $j > 0$, instead of $j < 0$, as will be seen in the case of the closure condition.

## SECTION 6

## INDICIAL EQUATION FOR THE DESCENDING SERIES SOLUTIONS

In the descending series solution, the highest power of $x$ which occurs is $k(n)$. This will be referred to as the order of the solution. Its relation to the parameter $n$ is determined by the indicial equation. We obtain this by first taking $\theta = 0$ in I-26 and then differentiating to get

$$\begin{aligned} y_n &= c_k^n x^k + \dots \\ y'_n &= k\, c_k^n x^{k-1} + \dots \\ y''_n &= k(k-1)\, c_k^n x^{k-2} + \dots \end{aligned} \qquad \text{I-34}$$

Substituting these results in I-28 we have

$$x^k c_k^n \left[ L_2(n) + B_2 k + A_2 k(k-1) \right] + x^{k+j}(\dots) + \dots \overset{(x)}{\equiv} 0$$

Hence the indicial equation for the order $k(n)$ of the descending series solutions is

$$A_2 k^2(n) + (B_2 - A_2)k(n) + L_2(n) = 0 \qquad \text{I-35}$$

Its solution for $L_2(n)$ is

$$L_2(n) = -A_2 k^2(n) - (B_2 - A_2)k(n) \qquad \text{I-36}$$

The solution of I-35 for $k(n)$ is

$$k(n) = \frac{-(B_2 - A_2) \pm \sqrt{(B_2 - A_2)^2 - 4A_2 L_2(n)}}{2A_2} \qquad \text{I-37}$$

## SECTION 7

## CLOSURE EQUATION FOR THE DESCENDING SERIES SOLUTIONS

By the term closed solution we mean a solution which terminates and is not an infinite series.

For closed solutions we must have $c^n_{k+\theta j} = 0$ for all $\theta$ greater than some particular value, say $\theta = \theta(n)$, a function of $n$. That is, $c^n_{k+\theta j} = 0$ for $\theta = \theta(n) + 1,\ \theta(n) + 2,$ ... . Hence in the recursion product I-30, that factor in the numerator obtained by using $\phi_{max} = \theta - 1$ must be zero for $\theta = \theta(n) + 1$; that is, for $\phi_{max} = [\theta(n) + 1] - 1 = \theta(n)$.

Accordingly, we have as a preliminary form of the condition for the closure of the descending series solutions

$$A_1[k+\theta(n)j]^2 + (B_1-A_1)[k+\theta(n)j] + L_1(n) = 0 \qquad \text{I-38}$$

But if $c^n_{k+\theta j} = 0$ for all $\theta > \theta(n)$, then $\theta(n)$ is the largest value of $\theta$ for which $c^n_{k+\theta j} \neq 0$, so that $k+\theta(n)j$ is the total subscript on the last non-vanishing coefficient in the descending series solution, and is therefore by I-26 the lowest power of x in the solution. This will be referred to as the subdegree of the solution. Denoting this by $h(n)$, we have

$$h(n) = k(n) + \theta(n)j \qquad \text{I-39}$$

Using this in I-38 we have the closure equation for the descending series solutions

$$A_1h^2(n) + (B_1-A_1)\,h(n) + L_1(n) = 0 \qquad \text{I-40}$$

## SECTION 8

## POLYNOMIAL SOLUTIONS OBTAINABLE FROM THE DESCENDING SERIES SOLUTIONS

Making use of the indicial and closure equations of the two preceding sections, we now consider the possibilities of obtaining polynomial solutions from the descending series solutions in the three cases discussed in section 3.

Case I: $L_1(n) = L_1$, a constant independent of n. I-16'

The indicial equation for the order $k(n)$ of the descending series solutions was

$$A_2k^2(n) + (B_2-A_2)\,k(n) + L_2(n) = 0 \qquad \text{I-35'}$$

This gives k(n) the order of the solution as a function of n. If $k(n)$ is assigned as say $k(n) = Nn$, N a positive integer, then I-35' determines the form of $L_2(n)$ as

$$L_2(n) = -A_2N^2n^2 - (B_2-A_2)\,Nn \qquad \text{I-41}$$

Using I-16' in I-40 we have the closure equation for the subdegree h of the descending series solutions as

$$A_1h^2 + (B_1-A_1)\,h + L_1 = 0 \qquad \text{I-42}$$

Since this equation is a quadratic in h, then h has at most two values. But from I-39 h is the total subscript on the last non-vanishing coefficient of the descending series solution. Hence, for $k(n) = Nn$, $j = -1$, which includes the classical Laguerre Polynomials (6, Chapter X), for which $N = 1$, then we must have $h = 0$, since we must have $k(n) \geqslant h$ for all n. Accordingly, I-42 tells us that we also have $L_1 = 0$ in this instance.

For $k(n) = Nn$, $j = -2$, which includes the classical Legendre, Hermite, and Jacobi polynomials (6, Chapters II, VIII, IX; 2, Chapter IX), for which $N = 1$, then we will have $h = 0, 1$, according as Nn is even or odd. Using $h = 0$ in I-42, we find that we must have $L_1 = 0$, as in the previous instance. Using $h = 1$, $L_1 = 0$ in I-42, we have

$$A_1 + (B_1 - A_1) + 0 = 0$$

from which we have $B_1 = 0$

Finally, we note that $|j| > 2$ requires more than two values of h, so that I-40 becomes an identity in h. Hence $A_1 = B_1 = L_1 = 0$, and the numerator of the recursion product I-30 vanishes for all n and all $\theta$, so that we then have the degenerate case of monomial solutions.

Case II: $L_2(n) = L_2$, a constant independent of n. I-19'

Using this in I-35 we have the indicial equation for the order $k$ of the descending series solutions as

$$A_2k^2 + (B_2 - A_2)\,k + L_2 = 0 \tag{I-43}$$

Since this is a quadratic in $k$, then $k$ has at most two values.

The closure equation for the subdegree $h(n)$ of the descending series solutions was

$$A_1h^2(n) + (B_1 - A_1)\,h(n) + L_1(n) = 0 \tag{I-40'}$$

This gives $h(n)$, the subdegree of the solutions, as a function of $n$. If $h(n)$ is assigned as say $h(n) = Gn$, $G$ a positive integer, then I-40' determines the form of $L_1(n)$ as

$$L_1(n) = -A_1G^2n^2 - (B_1 - A_1)\,Gn \tag{I-44}$$

Since from the definitions we must have $k \geq h(n) \geq 0$, then we see that this case will give us at most a finite set of polynomial solutions, and this only for $j = -1$ and $j = -2$, if $k$ and $h(n)$ are to be compatible with $j$, as we have seen from the discussion of case I.

Case III: $L_1(n) = K_1\, L(n), \quad L_2(n) = K_2\, L(n)$ . I-22'

where the K's are constants independent of n.

Using these in I-35 we have the indicial equation for the order $k(n)$ of the descending series solutions

$$A_2 k^2(n) + (B_2 - A_2)\, k(n) + K_2\, L(n) = 0 \qquad \text{I-45}$$

The solution of this for $k(n)$ is, using I-22' in I-37,

$$k(n) = \frac{-(B_2 - A_2) \pm \sqrt{(B_2 - A_2)^2 - 4A_2 K_2 L(n)}}{2A_2} \qquad \text{I-46}$$

Using I-22' in I-40, we have the closure equation for the subdegree $h(n)$ of the descending series solutions as

$$A_1 h^2(n) + (B_1 - A_1)\, h(n) + K_1\, L(n) = 0 \qquad \text{I-47}$$

We now assign $h(n)$ as

$$h(n) = Gn \qquad \text{I-48}$$

where G is a positive integer. Using this in I-47 and solving for the parameter L(n), we have

$$L(n) = \frac{-A_1G^2n^2 - (B_1-A_1)\ Gn}{K_1} \qquad \text{I-49}$$

Now let

$$K_2^* = \frac{K_2}{K_1} \qquad \text{I-50}$$

Using this and I-49 in I-46, we have

$$k(n) = \frac{-(B_2-A_2) \pm \sqrt{(B_2-A_2)^2 + 4A_2K_2^*\ A_1G^2n^2 + 4A_2K_2^*\ (B_1-A_1)\ Gn}}{2A_2}$$

Rearranging under the radical we have

$$k(n) = \frac{-(B_2-A_2) \pm \sqrt{4A_1A_2K_2^*\ G^2n^2 + 4A_2K_2^*\ (B_1-A_1)\ Gn + (B_2-A_2)^2}}{2A_2} \qquad \text{I-51}$$

For $k(n)$ to be rational for all $n$, the quantity under the radical must be a perfect square for all $n$. Hence we must have

$$4A_1A_2K_2^* G^2(B_2-A_2)^2 = 4A_2^2 K_2^{*2} G^2(B_1-A_1)^2$$

Rearranging and factoring, we have, after dividing out the 4,

$$G^2A_2K_2^*\left[A_1(B_2-A_2)^2 - A_2K_2^*(B_1-A_1)^2\right] = 0 \qquad \text{I-52}$$

This relation is satisfied if either $G$, $K_2^*$, the bracket, or $A_2$ is zero. These possibilities lead to the following results:

If $G = 0$, then from I-48 we have $h = 0$. Then from I-47 we have $K_1 L(n) = 0$. Hence, either $K_1 = 0$, or $L(n) = 0$.

If it is the $K_1$ which is zero when $G = h = 0$, then by I-22 and I-16 we have a special instance of case I with $L_1 = 0$.

If instead of the $K_1$ it is the $L(n)$ which is zero when $G = h = 0$, then by I-22 both $L_1(n)$ and $L_2(n)$ are zero, so that by I-1 the differential equation does not contain the term in $y_n$. Also, by I-45, we have $k = 0, \frac{A_2-B_2}{A_2}$, so that the differential equation has but two solutions.

If in I-52 the $K_2^* = 0$, then by I-50 the $K_2 = 0$. Then by I-22 and I-19 we have a special instance of case II with

$L_2 = 0$. Using $K_2 = 0$ in I-45, we again have that $k = 0$, $\frac{A_2-B_2}{A_2}$. Hence, one of the solutions discussed under case II is only a constant.

The bracket in I-52 being zero leads to the Non-Classical Orthogonal Polynomials discussed in the next part, Part II, of this dissertation.

The $A_2$ of I-52 being zero leads to the Non-Classical Orthogonal Polynomials discussed in Part III of this dissertation.

## SECTION 9

## INDICIAL EQUATION FOR THE ASCENDING SERIES SOLUTIONS

In the ascending series solutions the lowest power of x which occurs is $k(n)$. Hence, in accordance with the definition of section 7, $k(n)$ is here the subdegree of the solutions. Its relation to the parameter is determined by the indicial equation. We obtain this by using I-34 in I-32 to get

$$x^k c_k^n \left[L_1(n) + B_1 k + A_1 k(k-1)\right] + x^{k+j}(\ldots) + \ldots \overset{(x)}{\equiv} 0$$

Hence the indicial equation for the subdegree $k(n)$ of the ascending series solutions is

$$A_1 k^2(n) + (B_1 - A_1)\, k(n) + L_1(n) = 0 \qquad \text{I-53}$$

Its solution for $L_1(n)$ is

$$L_1(n) = -A_1 k^2(n) - (B_1 - A_1)\, k(n) \qquad \text{I-54}$$

The solution of I-53 for $k(n)$ is

$$k(n) = \frac{-(B_1 - A_1) \pm \sqrt{(B_1 - A_1)^2 - 4A_1 L_1(n)}}{2A_1} \qquad \text{I-55}$$

We observe that I-53, the indicial equation for the subdegree $k(n)$ of the ascending series solutions, is the same as I-40, the closure equation for the subdegree $h(n)$ of the descending series solutions, except that here we have k instead h.

## SECTION 10

## CLOSURE EQUATION FOR THE ASCENDING SERIES SOLUTIONS

For closed solutions we must have $c^n_{k+\theta j} = 0$ for all $\theta$ greater than some particular value, say $\theta = \theta(n)$, a function of n. That is, $c^n_{k+\theta j} = 0$ for $\theta = \theta(n) + 1, \ \theta(n) + 2, \ldots$ Hence in the recursion product I-33 that factor in the numerator obtained by using $\phi_{max} = \theta - 1$ must be zero for $\theta = \theta(n) + 1$; that is, for $\phi_{max} = [\theta(n) + 1] - 1 = \theta(n)$.

Accordingly, we have as a preliminary form of the condition for the closure of the ascending series solutions

$$A_2[k+\theta(n)j]^2 + (B_2-A_2)[k+\theta(n)j] + L_2(n) = 0 \qquad \text{I-56}$$

But if $c^n_{k+\theta j} = 0$ for all $\theta > \theta(n)$, then $\theta(n)$ is the largest value of $\theta$ for which $c^n_{k+\theta j} \neq 0$, so that $k+\theta(n)j$ is the total subscript on the last non-vanishing coefficient of the ascending series solution, and is therefore by the definition of section 6 the order of the solution. Denoting this by $h(n)$, we have

$$h(n) = k(n) + \theta(n)j \qquad \text{I-39'}$$

the same relation as was obtained for the descending series solutions, except that the roles of h and k have been interchanged. Using this in I-56, we have the closure equation for the ascending series solutions

$$A_2h^2(n) + (B_2-A_2)\ h(n) + L_2(n) = 0 \qquad \text{I-57}$$

We observe that this the closure equation for the order h(n) of the ascending series solutions is the same as I-35, the indicial equation for the order k(n) of the descending series solutions, the roles of h(n) and k(n) being interchanged.

## SECTION 11

## POLYNOMIAL SOLUTIONS OBTAINABLE FROM THE ASCENDING SERIES SOLUTIONS

Making use of the indicial and closure equations of the two preceding sections, we now consider the possibilities of obtaining polynomial solutions from the ascending series solutions in the three cases discussed in section 3.

Case I: $L_1(n) = L_1$, a constant independent of n. I-16"

The closure equation for the order h(n) of the ascending series solutions was

$$A_2 h^2(n) + (B_2 - A_2)\, h(n) + L_2(n) = 0 \qquad \text{I-57'}$$

which is the same as I-35' (ff. I-40), the indicial equation for the order $k(n)$ of the descending series solutions, except that here we have h instead of k. Hence our conclusions are the same as those following I-35', q.v.

Using I-16" in I-53, we have the indicial equation for the subdegree k of the ascending series solutions

$$A_1 k^2 + (B_1 - A_1)\, k + L_1 = 0 \qquad \text{I-58}$$

which is the same as I-42, the closure equation for the subdegree h of the descending series solutions, except that here we have k instead of h. Hence our conclusions are the same as those following I-42, q.v.

Case II: $L_2(n) = L_2$, a constant independent of n. I-19"

Using this in I-57 we have the closure equation for the order h of the ascending series solutions

$$A_2h^2 + (B_2-A_2)\ h + L_2 = 0 \qquad \text{I-59}$$

which is the same as I-43, the indicial equation for the order $k$ of the descending series solutions, except that here we have h instead of k. Hence our conclusions are the same as those following I-43, q.v.

The indicial equation for the subdegree k(n) of the ascending series solutions was

$$A_1k^2(n) + (B_1-A_1)\ k(n) + L_1(n) = 0 \qquad \text{I-53'}$$

which is the same as I-40' (ff.I-43), the closure equation for the subdegree h(n) of the descending series solutions, except that here we have k instead of h. Hence our conclusions here are the same as those following I-40', q.v.

Case III: $L_1(n) = K_1\ L(n),\ \ L_2(n) = K_2\ L(n)$ I-22"

Using these in I-57, we have the closure equation for the order h(n) of the ascending series solutions

$$A_2h^2(n) + (B_2-A_2)\ h(n) + K_2\ L(n) = 0 \qquad \text{I-60}$$

which is the same as I-45, the indicial equation for the order $k(n)$ of the descending series solutions, except that here we have h instead of k.

Using I-22" in I-53, we have the indicial equation for the subdegree $k(n)$ of the ascending series solutions

$$A_1 k^2(n) + (B_1 - A_1)\, k(n) + K_1\, L(n) = 0 \qquad \text{I-61}$$

which is the same as I-47, the closure equation for the subdegree $h(n)$ of the descending series solutions, except that here we have k instead of h.

If we now proceed with these two equations I-60 and I-61 as we did with I-45 and I-47 of the descending series solutions, the roles of h and k being interchanged, so that I-48 becomes

$$k(n) = Gn \qquad \text{I-62}$$

then our conclusions will be similar to those following I-45 and I-47, q.v.

## SECTION 12

## SOLUTIONS OF THE DEGENERATE FIRST ORDER EQUATION, CASE I

From I-1 the differential equation degenerates to first order when

$$A_1 = A_2 = 0 \qquad \text{I-63}$$

and from section 3 we had

Case I: $L_1(n) = L_1$, a constant independent of $n$. I-16'''

Using these results in I-35' (ff.I-40), we have the indicial equation for the order $k(n)$ of the descending series solutions

$$B_2 k(n) + L_2(n) = 0 \qquad \text{I-64}$$

from which we have

$$L_2(n) = -B_2\ k(n) \qquad \text{I-65}$$

Using I-63 in I-42, we have the closure equation for the subdegree h of the descending series solutions

$$B_1 h + L_1 = 0 \qquad \text{I-66}$$

from which we have

$$L_1 = -B_1 h \qquad \text{I-67}$$

and

$$h = -\frac{L_1}{B_1} \qquad \text{I-68}$$

Using I-63, I-16"', I-65, and I-67 in I-30, we have the recursion product for the descending series solutions

$$c^n_{k+\theta j} = \left(-\frac{B_1}{B_2 j}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} j\phi + k(n) - h}{\theta!} c^n_k \qquad \text{I-69}$$

which gives an infinite set of polynomial solutions.

From this we see that if $B_1 = 0$, then we have monomial solutions; also, I-67 gives $L_1 = 0$.

Using I-63 in I-57', we have the closure equation for the order $h(n)$ of the ascending series solutions

$$B_2 h(n) + L_2(n) = 0 \qquad \text{I-70}$$

from which we have

$$L_2(n) = -B_2\, h(n) \tag{I-71}$$

Using I-63 in I-58, we have the indicial equation for the subdegree k of the ascending series solutions

$$B_1 k + L_1 = 0 \tag{I-72}$$

from which we have

$$L_1 = -B_1 k \tag{I-73}$$

and

$$k = -\frac{L_1}{B_1} \tag{I-74}$$

Using I-63, I-16"', I-71, and I-73 in I-33, we have the recursion product for the ascending series solutions

$$c^n_{k+\theta j} = \left(-\frac{B_2}{B_1 j}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} j\phi + k - h(n)}{\theta!}\, c^n_k \tag{I-75}$$

which again gives an infinite set of polynomial solutions.

From this we see that if $B_2 = 0$, then we have monomial solutions; also, I-71 gives $L_2(n) = 0$, so that the parameter is gone from the differential equation.

## SECTION 13

## ORTHOGONALITY OF THE SOLUTIONS OF THE DEGENERATE FIRST ORDER EQUATION, CASE I

Let

$$\bar{w} = \text{an integrating factor} \qquad \text{I-76}$$

Using I-63 and I-16"' in I-1, multiplying two such equations by $\bar{w}$ and each equation by the solution of the other, we have by addition

$$\begin{array}{c|l} y_m & \bar{w}(B_1x^{r+1} + B_2x^{s+1})y_n' + \bar{w}\left[L_1x^r + L_2(n)x^s\right]y_n = 0 \\ y_n & \bar{w}(B_1x^{r+1} + B_2x^{s+1})y_m' + \bar{w}\left[L_1x^r + L_2(m)x^s\right]y_m = 0 \\ \hline \end{array}$$

$$\bar{w}(B_1x^{r+1} + B_2x^{s+1})(y_my_n' + y_ny_m') + 2L_1\bar{w}x^r(y_my_n)$$

$$+ \left[L_2(m) + L_2(n)\right]\bar{w}x^s\, y_my_n = 0$$

But we observe that

$$\frac{d}{dx}\left[\bar{w}(B_1x^{r+1}+B_2x^{s+1})(y_my_n)\right] = \bar{w}(B_1x^{r+1}+B_2x^{s+1})(y_my_n'+y_ny_m')$$

$$+\frac{d}{dx}\left[\bar{w}(B_1x^{r+1}+B_2x^{s+1})\right](y_my_n) \qquad \text{I-77}$$

Accordingly, integration of our preceding result gives

$$\left[\bar{w}(B_1x^{r+1}+B_2x^{s+1})y_my_n\right]_{x_0}^{x_1}+\left[L_2(m)+L_2(n)\right]\int_{x_0}^{x_1}\bar{w}x^s\;y_my_n\;dx = 0 \qquad \text{I-78}$$

provided that

$$\frac{d}{dx}\left[\bar{w}(B_1x^{r+1}+B_2x^{s+1})\right] = 2L_1\bar{w}x^r \qquad \text{I-79}$$

Dividing by $\bar{w}(B_1x^{r+1}+B_2x^{s+1})$ and multiplying by dx, we have, using I-3,

$$\frac{d(\bar{w}q)}{\bar{w}q} = \frac{2L_1x^r\;dx}{B_1x^{r+1}+B_2x^{s+1}}$$

Dividing our an $x^r$ in the right member and integrating, we have

$$\ln\left(\overline{w}q\right) = 2L_1 \int \frac{dx}{B_1x + B_2x^{s-r+1}} \qquad \text{I-80}$$

Multiplying by $\frac{B_1}{B_1}$ and adding and subtracting $\frac{B_2x^{s-r}}{B_1x + B_2x^{s-r+1}}$

we have

$$\ln\left(\overline{w}q\right) = \frac{2L_1}{B_1}\left[\int \frac{\left(B_1 + B_2x^{s-r}\right)dx}{B_1x + B_2x^{s-r+1}} - \int \frac{B_2x^{s-r}\,dx}{B_1x + B_2x^{s-r+1}}\right]$$

$$= \frac{2L_1}{B_1}\left[\int \frac{dx}{x} - \frac{1}{s-r}\int \frac{(s-r)B_2x^{s-r-1}\,dx}{B_1 + B_2x^{s-r}}\right]$$

$$= \frac{2L_1}{B_1}\left[\ln x - \frac{1}{s-r}\ln\left(B_1 + B_2x^{s-r}\right)\right]$$

$$= \frac{2L_1}{B_1}\ln\left[x\left(B_1 + B_2x^{s-r}\right)^{\frac{1}{r-s}}\right]$$

$$\overline{w}q = \left[x\left(B_1 + B_2x^{s-r}\right)^{\frac{1}{r-s}}\right]^{\frac{2L_1}{B_1}}$$

$$\overline{w}q = x^{\frac{2L_1}{B_1}} (B_1 + B_2x^{s-r})^{\frac{2L_1}{B_1(r-s)}} \qquad \text{I-81}$$

Using I-3 again, we have

$$\overline{w} = x^{\frac{2L_1}{B_1} - r - 1} (B_1 + B_2x^{s-r})^{\frac{2L_1}{B_1(r-s)} - 1} \qquad \text{I-82}$$

Now for polynomials the subdegree is $\geq 0$. Hence from I-68 or I-74, $\frac{L_1}{B_1} \leq 0$. But, as in I-27 or I-31, we take $s > r$, so that $r-s < 0$. Hence the exponent on the parentheses in I-81 is $\geq 0$. By I-68 or I-74 the subdegree of the product

$$(x^{\frac{2L_1}{B_1}} y_m y_n)$$

which is in the bracket of I-78 by virtue of I-3 and I-81, is zero. Therefore the bracket of I-78 vanishes at the roots of

$$(B_1 + B_2x^{s-r})$$

and if $|s-r| = |j|$ is even, $\frac{B_1}{B_2} < 0$, and $L_2(m) \neq L_2(n)$, then we have a real interval of orthogonality, and, again from I-78, the orthogonality relation

$$\int_{x_0}^{x_1} \overline{w}x^s y_m y_n \, dx = 0, \quad m \neq n \qquad \text{I-83}$$

If $B_1 = 0$, as well as $A_1 = A_2 = 0$ of I-63, then I-80 becomes

$$\ln(\overline{w}q) = 2L_1 \int \frac{dx}{B_2 x^{s-r+1}}$$

$$= \frac{2L_1}{B_2} \int x^{r-s-1} \, dx$$

$$= \frac{2L_1 x^{r-s}}{B_2(r-s)}$$

$$\overline{w}q = e^{\frac{2L_1 x^{r-s}}{B_2(r-s)}} \qquad \text{I-84}$$

But for $B_1 = 0$, then I-66 or I-72 gives $L_1 = 0$. Hence I-84 gives $\overline{w}q = 1$, so that there is no interval of orthogonality.

If $B_2 = 0$, as well as $A_1 = A_2 = 0$ of I-63, then I-80 becomes

$$\ln (\overline{w}q) = 2L_1 \int \frac{dx}{B_1 x}$$

$$= \frac{2L_1}{B_1} \ln x$$

$$= \ln x^{\frac{2L_1}{B_1}}$$

$$\overline{w}q = x^{\frac{2L_1}{B_1}} \qquad \text{I-85}$$

Since $\frac{L_1}{B_1} < 0$, as shown ff.I-82, then there are no finite roots of I-85, and therefore no interval of orthogonality.

## SECTION 14

## VANISHING OF THE INTEGRATED SQUARE OF THE SOLUTIONS OF THE DEGENERATE FIRST ORDER EQUATION, CASE I

Using I-63 in I-16"' in I-1, and multiplying the result by $2\bar{w}y_n$, we have

$$2\bar{w}(B_1x^{r+1} + B_2x^{s+1})y_ny_n' + 2\bar{w}\left[L_1x^r + L_2(n)x^s\right]y_n^2 = 0$$

Using I-79 and the relation

$$\frac{d}{dx}(y_n^2) = 2y_ny_n' \tag{I-86}$$

in this, we have by integration

$$\left[\bar{w}(B_1x^{r+1}+B_2x^{s+1})y_n^2\right]_{x_0}^{x_1} + 2L_2(n)\int_{x_0}^{x_1}\bar{w}x^s\,y_n^2\,dx = 0$$

Using I-3 and I-81 in the bracket, we have

$$\left[x^{\frac{2L_1}{B_1}}(B_1 + B_2x^{s-r})^{\frac{2L_1}{B_1(r-s)}}\,y_n^2\right]_{x_0}^{x_1} + 2L_2(n)\int_{x_0}^{x_1}\bar{w}x^s\,y_n^2\,dx = 0$$

The bracket vanishes under the same conditions as for the establishment of the orthogonality relation I-83, so that we have

$$\int_{x_0}^{x_1} \bar{w} x^s y_n^2 \, dx = 0, \quad L_2(n) \neq 0 \qquad \text{I-87}$$

That is, the integrated square of the solutions vanishes for all n such that $L_2(n) \neq 0$. Hence we cannot expand an arbitrary function in terms of the solutions.

## SECTION 15

## SOLUTIONS OF THE DEGENERATE FIRST ORDER EQUATION, CASE II

From I-1 the differential equation degenerates to first order when

$$A_1 = A_2 = 0 \qquad \text{I-63'}$$

and from section 3 we had

Case II: $L_2(n) = L_2$, a constant independent of n. I-19"'

Using I-63' in I-43, we have the indicial equation for the order k of the descending series solutions

$$B_2 k + L_2 = 0 \qquad \text{I-88}$$

from which we have

$$L_2 = -B_2 k \qquad \text{I-89}$$

and

$$k = -\frac{L_2}{B_2} \qquad \text{I-90}$$

Using I-63' in I-40', we have the closure equation for the subdegree h(n) of the descending series solutions

$$B_1\, h(n) + L_1(n) = 0 \qquad \text{I-91}$$

from which we have

$$L_1(n) = -B_1\, h(n) \qquad \text{I-92}$$

Using I-63', I-19"', I-89, and I-92 in I-30, we have the recursion product for the descending series solutions

$$c^{n}_{k+\theta j} = \left(-\frac{B_1}{B_2 j}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} j\phi + k - h(n)}{\theta!} c^{n}_{k} \qquad \text{I-93}$$

which gives us only a finite number of polynomial solutions.

From this we see that if $B_1 = 0$, then we have monomial solutions; also, I-91 gives $L_1(n) = 0$, so that the parameter is gone from the differential equation.

Using I-63' in I-59, we have the closure equation for the order h of the ascending series solutions

$$B_2 h + L_2 = 0 \qquad \text{I-94}$$

from which we have

$$L_2 = - B_2 h \qquad \text{I-95}$$

and

$$h = - \frac{L_2}{B_2} \qquad \text{I-96}$$

Using I-63' in I-53', we have the indicial equation for the subdegree $k(n)$ of the ascending series

$$B_1 \, k(n) + L_1(n) = 0 \qquad \text{I-97}$$

from which we have

$$L_1(n) = -B_1 \, k(n) \qquad \text{I-98}$$

Using I-63', I-19"', I-95, and I-98 in I-33, we have the recursion product for the ascending series solutions

$$c^n_{k+\theta j} = \left( - \frac{B_2}{B_1 j} \right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} j\phi + k(n) - h}{\theta!} \, c^n_k \qquad \text{I-99}$$

which again gives us only a finite number of polynomial solutions.

From this we see that if $B_2 = 0$, then we have monomial solutions; also, I-94 gives $L_2 = 0$.

## SECTION 16

## ORTHOGONALITY OF THE SOLUTIONS
## OF THE DEGENERATE FIRST ORDER EQUATION, CASE II

Using I-53' and I-19"' in I-1, multiplying two such equations by the $\overline{w}$ of I-76, and each equation by the solution of the other, we have by addition

$$\begin{array}{c|l} y_m & \overline{w}(B_1x^{r+1} + B_2x^{s+1})y_n' + \overline{w}\left[L_1(n)x^r + L_2x^s\right]y_n = 0 \\ y_n & \overline{w}(B_1x^{r+1} + B_2x^{s+1})y_m' + \overline{w}\left[L_1(m)x^r + L_2x^s\right]y_m = 0 \\ \hline \end{array}$$

$$\overline{w}(B_1x^{r+1} + B_2x^{s+1})(y_my_n' + y_ny_m') + 2L_2\overline{w}x^s(y_my_n)$$

$$+ \left[L_1(n) + L_1(m)\right]\overline{w}x^r\ y_my_n = 0$$

Using I-77, we have by integration of this

$$\left[\overline{w}(B_1x^{r+1} + B_2x^{s+1})y_my_n\right]_{x_0}^{x_1} + \left[L_1(m) + L_1(n)\right]\int_{x_0}^{x_1} \overline{w}x^r\ y_my_n\ dx = 0 \qquad \text{I-100}$$

provided that

$$\frac{d}{dx}\left[\overline{w}(B_1x^{r+1} + B_2x^{s+1})\right] = 2L_2\overline{w}x^s \qquad \text{I-101}$$

Dividing by $\overline{w}(B_1x^{r+1} + B_2x^{s+1})$ and multiplying by dx, we have, using I-3,

$$\frac{d(\overline{w}q)}{\overline{w}q} = \frac{2L_2x^s\,dx}{B_1x^{r+1} + B_2x^{s+1}}$$

Dividing out an $x^s$ in the right member and integrating, we have

$$\ln\,(\overline{w}q) = 2L_2 \int \frac{dx}{B_1x^{r-s+1} + B_2x} \qquad \text{I-102}$$

Multiplying by $\frac{B_2}{B_2}$ and adding and subtracting $\frac{B_1x^{r-s}}{B_1x^{r-s+1} + B_2x}$ we have

$$\ln\,(\overline{w}q) = \frac{2L_2}{B_2}\left[\int \frac{(B_1x^{r-s} + B_2)\,dx}{B_1x^{r-s+1} + B_2x} - \int \frac{B_1x^{r-s}\,dx}{B_1x^{r-s+1} + B_2x}\right]$$

$$= \frac{2L_2}{B_2}\left[\int \frac{dx}{x} - \frac{1}{r-s}\int \frac{(r-s)B_1x^{r-s-1}\,dx}{B_1x^{r-s} + B_2}\right]$$

$$= \frac{2L_2}{B_2}\left[\ln x - \frac{1}{r-s}\ln\left(B_1x^{r-s} + B_2\right)\right]$$

$$= \frac{2L_2}{B_2}\ln\left[x\left(B_1x^{r-s} + B_2\right)^{\frac{1}{s-r}}\right]$$

$$\overline{w}q = \left[x\left(B_1x^{r-s} + B_2\right)^{\frac{1}{s-r}}\right]^{\frac{2L_2}{B_2}}$$

$$= x^{\frac{2L_2}{B_2}}\left(B_1x^{r-s} + B_2\right)^{\frac{2L_2}{B_2(s-r)}} \qquad \text{I-103}$$

Using I-3 again, we have

$$\overline{w} = x^{\frac{2L_2}{B_2} - s - 1}\left(B_1x^{r-s} + B_2\right)^{\frac{2L_2}{B_2(s-r)} - 1} \qquad \text{I-104}$$

We may write I-103 as

$$\overline{w}q = \left[x^{s-r}\left(B_1x^{r-s} + B_2\right)\right]^{\frac{2L_2}{B_2(s-r)}}$$

from which we have

$$\bar{w}q = (B_1 + B_2 x^{s-r})^{\frac{2L_2}{B_2(s-r)}} \qquad \text{I-105}$$

Now for polynomials the order is $\geq 0$, and in this case II it is determined by I-90 or I-96, which then give $\frac{L_2}{B_2} < 0$. But, as in I-27 or I-31, we take $s > r$, so that $s-r > 0$. Hence the exponent on the parentheses in I-105 is negative, so that the factor is essentially in the denominator. Accordingly, using I-3, the bracket in I-100 becomes, with the aid of I-90 or I-96, and I-92 or I-98,

$$\bar{w}q\, y_m y_n = \frac{c_k^m c_k^n x^{-\frac{2L_2}{B_2}} + \dots + c_{h(m)}^m c_{h(n)}^n x^{h(m)+h(n)}}{B_2^{-\frac{2L_2}{B_2(s-r)}} x^{-\frac{2L_2}{B_2}} + \dots + B_1^{-\frac{2L_2}{B_2(s-r)}} x^0}$$

of which we can say only that it vanishes only at the origin, provided that $h(m) + h(n) > 0$. This means that in general there can be no interval of orthogonality.

If $B_1 = 0$, as well as $A_1 = A_2 = 0$ of I-63, then I-102 becomes

$$\ln(\overline{w}q) = 2L_2 \int \frac{dx}{B_2 x}$$

$$= \frac{2L_2}{B_2} \ln x$$

$$= \ln x^{\frac{2L_2}{B_2}}$$

$$\overline{w}q = x^{\frac{2L_2}{B_2}} \qquad \text{I-106}$$

Since $\frac{L_2}{B_2} < 0$, as shown ff. I-105, then there are no finite roots of I-106, and therefore no interval of orthogonality.

If $B_2 = 0$, as well as $A_1 = A_2 = 0$ of I-63, then I-102 becomes

$$\ln(\overline{w}q) = 2L_2 \int \frac{dx}{B_1 x^{r-s+1}}$$

$$\ln(\overline{w}q) = \frac{2L_2}{B_1} \int x^{s-r-1}\, dx$$

$$= \frac{2L_2 x^{s-r}}{B_1(s-r)}$$

$$\overline{w}q = e^{\frac{2L_2 x^{s-r}}{B_1(s-r)}} \qquad \text{I-107}$$

But for $B_2 = 0$, then I-88 or I-94 gives $L_2 = 0$. Hence I-107 gives $\overline{w}q = 1$, so that there is no interval of orthogonality.

## SECTION 17

## VANISHING OF THE INTEGRATED SQUARE OF THE SOLUTIONS OF THE DEGENERATE FIRST ORDER EQUATION, CASE II

Using I-63' and I-19"' in I-1, and multiplying the result by $2\overline{w}y_n$, we have

$$2\overline{w}(B_1 x^{r+1} + B_2 x^{s+1}) y_n y_n' + 2\overline{w}\left[L_1(n)x^r + L_2 x^s\right] y_n^2 = 0$$

Using I-101 and I-86 in this, we have by integration

$$\left[\bar{w}(B_1 x^{r+1} + B_2 x^{s+1}) y_n^2\right]_{x_0}^{x_1} + 2L_1(n) \int_{x_0}^{x_1} \bar{w} x^r y_r^2 \, dx = 0 \qquad \text{I-108}$$

The bracket here is the same as that in I-100 for which we have shown there is no interval of orthogonality.

## SECTION 18

## NON-TELESCOPING OF THE ASCENDING SERIES SOLUTIONS, CASE I

Using I-16 in I-33, we have the recursion product for the ascending series solutions

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_2(k+\phi j)^2 + (B_2-A_2)(k+\phi j) + L_2(n)}{\prod_{\phi=1}^{\theta} A_1(k+\phi j)^2 + (B_1-A_1)(k+\phi j) + L_1} c^n_k \qquad \text{I-109}$$

The indicial equation for the subdegree k of the ascending series solutions, I-58, shows that k is a constant having at most two values. Hence the denominator of I-109 does not contain n, whereas the numerator does , and only in the term $L_2(n)$. Therefore, there can be no cancellation of factors holding for all n in the recursion product for the ascending series solutions.

## SECTION 19

## NON-TELESCOPING OF THE DESCENDING SERIES SOLUTIONS, CASE I

Using I-16 and I-36 in I-30, we have the recursion product for the descending series solutions

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_1(k+\phi j)^2 + (B_1-A_1)(k+\phi j) + L_1}{\prod_{\phi=1}^{\theta} A_2(k+\phi j)^2 + (B_2-A_2)(k+\phi j) - A_2k^2 - (B_2-A_2)k}\, c^n_k$$

$$= (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_1 j^2\phi^2 + 2A_1 j\phi k + A_1 k^2 + j(B_1-A_1)\phi + (B_1-A_1)k + L_1}{\prod_{\phi=1}^{\theta} A_2 j^2\phi^2 + 2A_2 j\phi k + j(B_2-A_2)\phi}\, c^n_k$$

$$= \left(\frac{-1}{j}\right)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_1 j^2\phi^2 + 2A_1 j\phi k + A_1 k^2 + j(B_1-A_1)\phi + (B_1-A_1)k + L_1}{\prod_{\phi=1}^{\theta} \phi\,(A_2 j\phi + 2A_2 k + B_2 - A_2)}\, c^n_k \tag{I-110}$$

Put $(\phi+i)$ for $\phi$ in the denominator and obtain

$$c^n_{k+\theta j} = \left(\frac{-1}{j}\right)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_1 j^2 \phi^2 + 2A_1 j\phi k + A_1 k^2 + j(B_1-A_1)\phi + (B_1-A_1)k + L_1}{\prod_{\phi=1-i}^{\theta-i} (\phi+i)\left[A_2 j\phi + 2A_2 k + (A_2 ji + B_2 - A_2)\right]} c^n_k \qquad \text{I-111}$$

For $(\phi+i)$ to be a factor of the numerator for all $n$ for some particular value of $i$, then $\phi = -i$ must make the numerator identically equal to zero for all values of $k$, since from I-35', we have $k = k(n)$, a function of $n$. That is, we must have

$$A_1 j^2(-i)^2 + 2A_1 j(-i)k + A_1 k^2 + j(B_1-A_1)(-i) + (B_1-A_1)k + L_1 \overset{(k)}{\equiv} 0$$

Therefore the coefficients of the $k$'s must all be zero. Hence we have the equations:

$k^2$ : $$A_1 = 0 \qquad \text{I-112}$$

$k$ : $$-2A_1 ji + B_1 - A_1 = 0 \qquad \text{I-113}$$

$1$ : $$A_1 j^2 i^2 - j(B_1 - A_1)i + L_1 = 0 \qquad \text{I-114}$$

From I-112 we have $A_1 = 0$. Using this in I-113 gives us $B_1 = 0$. Then I-114 gives us $L_1 = 0$. Hence, from I-110 the numerator of the recursion product is zero, and we have monomial solutions.

For j to be a factor of the numerator of I-110 for all n requires that the coefficients of the terms not containing j shall be zero. This requirement gives us the equations

$k^2$: $$A_1 = 0$$

k : $$B_1 - A_1 = 0$$

1 : $$L_1 = 0$$

Using the $A_1 = 0$ from the first of these in the second gives $B_1 = 0$. Thus again the numerator of the recursion product I-110 is zero and we have monomial solutions.

However, $A_1$, $B_1$, $L_1$ may all be divisible by j, since j is a constant, and in this way j may be a factor of the numerator of I-110.

If now for all n and some particular value of i the numerator of I-111 is to have in common with the denominator the factor in the bracket of the denominator, then we must have

$$A_1 j^2 \phi^2 + 2A_1 j\phi k + A_1 k^2 + j(B_1 - A_1)\phi + (B_1 - A_1)k + L_1$$

$$\overset{(\phi,k)}{\equiv} (E\phi + Fk + H)\left[A_2 j\phi + 2A_2 k + (A_2 ji + B_2 - A_2)\right] \qquad \text{I-115}$$

Therefore the coefficients of the corresponding powers of $\phi$ and $k$ must be equal. Hence we have the equations

$\phi^2$: $$A_1 j^2 = A_2 jE \qquad \text{I-116}$$

$\phi k$: $$2A_1 j = 2A_2 E + A_2 jF \qquad \text{I-117}$$

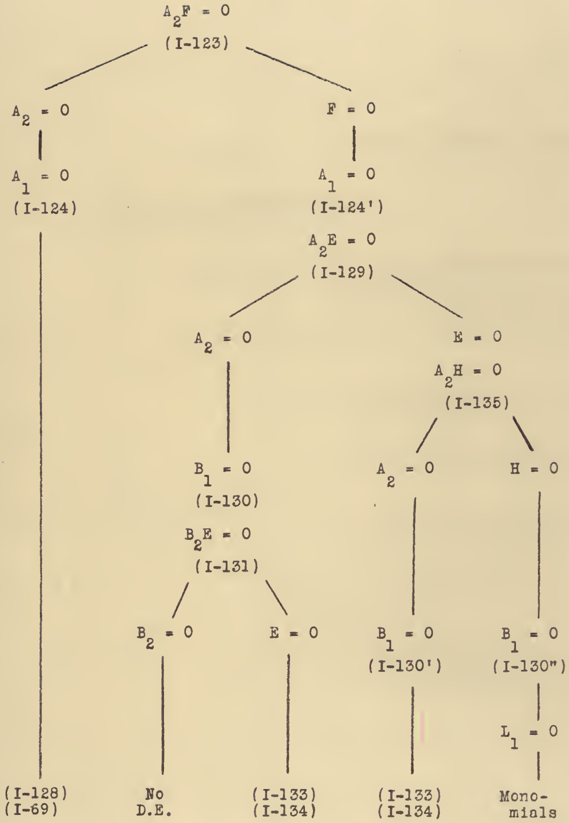
$k^2$: $$A_1 = 2A_2 F \qquad \text{I-118}$$

$\phi$ : $$j(B_1 - A_1) = (A_2 ji + B_2 - A_2)E + A_2 jH \qquad \text{I-119}$$

$k$ : $$(B_1 - A_1) = (A_2 ji + B_2 - A_2)F + 2A_2 H \qquad \text{I-120}$$

$1$ : $$L_1 = (A_2 ji + B_2 - A_2)H \qquad \text{I-121}$$

We proceed to show that these equations are satisfied in five ways yielding four different results, as summarized in the following chart:


$A_2F = 0$
(I-123)
$A_2 = 0$
$F = 0$
$A_1 = 0$
(I-124)
$A_1 = 0$
(I-124')
$A_2E = 0$
(I-129)
$A_2 = 0$
$E = 0$
$A_2H = 0$
(I-135)
$B_1 = 0$
(I-130)
$A_2 = 0$
$H = 0$
$B_2E = 0$
(I-131)
$B_2 = 0$
$E = 0$
$B_1 = 0$
(I-130')
$B_1 = 0$
(I-130")
$L_1 = 0$
(I-128)
(I-69)
No
D.E.
(I-133)
(I-134)
(I-133)
(I-134)
Mono-
mials

I-122

From I-116, either $E = \frac{A_1 j}{A_2}$, or $j = o$. The latter we dismiss as giving monomial solutions. Using the former in I-117 gives

$$2A_1 j = 2A_2 (\frac{A_1 j}{A_2}) + A_2 jF$$

$$= 2A_1 j + A_2 jF$$

Hence

$$A_2 jF = 0$$

Since for other than monomial solutions we must have $j \neq o$, then

$$A_2 F = 0 \qquad \text{I-123}$$

If $A_2 = 0$ from this, then I-116 gives

$$A_1 = 0 \qquad \text{I-124}$$

With these I-118 is satisfied. Also, I-119 gives

$$jB_1 = B_2 E$$

from which

$$E = \frac{B_1 j}{B_2} \qquad \text{I-125}$$

I-120 gives

$$B_1 = B_2 F$$

from which

$$F = \frac{B_1}{B_2} \qquad \text{I-126}$$

I-121 gives

$$L_1 = B_2 H$$

from which

$$H = \frac{L_1}{B_2} \qquad \text{I-127}$$

Accordingly, using $A_2 = A_1 = 0$ from I-123 and I-124, and the results I-125, I-126, I-127 in I-115, we have

$$B_1 j\phi + B_1 k + L_1 = \left(\frac{B_1 j}{B_2}\phi + \frac{B_1}{B_2}k + \frac{L_1}{B_2}\right)\left[B_2\right] \qquad \text{I-128}$$

Thus the common factor, the bracket of I-115, is reduced to the constant $B_2$. Moreover, with $A_1 = A_2 = 0$, the differential equation reduces to first order, with the results consequent to I-63.

If $F = 0$ from I-123, then I-118 gives

$$A_1 = 0 \tag{I-124'}$$

With this in I-116, we have, since $j \neq 0$,

$$A_2 E = 0 \tag{I-129}$$

If $F = A_1 = A_2 = 0$ from I-123, I-124', and I-129, respectively, then I-120 gives

$$B_1 = 0 \tag{I-130}$$

Using these results in I-119, we have

$$B_2 E = 0 \tag{I-131}$$

If $F = A_1 = A_2 = B_1 = B_2 = 0$, from I-123, I-124', I-129, I-130, and I-131, respectively, then from I-1 we see that the differential equation no longer exists.

If $F = A_1 = A_2 = B_1 = E = 0$, from I-123, I-124', I-129, I-130, and I-131, respectively, then I-121 gives

$$L_1 = B_2 H$$

from which

$$H = \frac{L_1}{B_2} \tag{I-132}$$

Using these results in I-115, we have

$$L_1 = \left(\frac{L_1}{B_2}\right)\left[B_2\right] \tag{I-133}$$

Thus the common factor, the bracket of I-115, is again reduced to the constant $B_2$. Moreover, with $A_1 = A_2 = 0$, the differential equation is again reduced to first order, with results consequent to I-63, except that in this instance the recursion product I-111 reduces to

$$c^n_{k+\theta j} = \left(-\frac{L_1}{B_2 j}\right)^{\theta} \frac{1}{\theta!}\, c^n_k \tag{I-134}$$

instead of to I-69, so that we have an infinite series solution provided $L_1 \neq 0$, in which case we have monomials, as followed also from I-69.

If $F = A_1 = E = 0$, from I-123, I-124', I-129, respectively, then I-119 gives

$$B_1 j = A_2 jH$$

from which

$$B_1 = A_2 H$$

Also, I-120 gives

$$B_1 = 2A_2 H$$

Equating these two expressions for $B_1$ gives us

$$2A_2 H = A_2 H$$

from which

$$A_2 H = 0 \qquad \text{I-135}$$

If $F = A_1 = E = A_2 = 0$, from I-123, I-124', I-129, and I-135, respectively, then I-119 or I-120 gives

$$B_1 = 0 \qquad \text{I-130'}$$

Also, I-121 gives

$$L_1 = B_2 H$$

from which

$$H = \frac{L_1}{B_2} \qquad \text{I-132'}$$

Accordingly, we again have the results I-133 and I-134.

If $F = A_1 = E = H = 0$, from I-123, I-124', I-129, and I-135, respectively, then I-119 or I-120 gives

$$B_1 = 0 \qquad \text{I-130"}$$

and I-121 gives

$$L_1 = 0 \qquad \text{I-136}$$

With these results the recursion product I-111 becomes zero in the numerator, and we have monomial solutions.

## SECTION 20

## NON-TELESCOPING OF THE DESCENDING SERIES SOLUTIONS, CASE II

Using I-19 in I-30 we have the recursion product for the descending series solutions

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_1(k+\phi j)^2 + (B_1-A_1)(k+\phi j) + L_1(n)}{\prod_{\phi=1}^{\theta} A_2(k+\phi j)^2 + (B_2-A_2)(k+\phi j) + L_2} c^n_k \qquad \text{I-137}$$

The indicial equation for the order k of the descending series solutions, I-43, shows that k is a constant having at most two values.

Thus the situation here in I-137 is the same as that in I-109, the recursion product for the ascending series solutions, Case I. In fact, a mere interchange of subscripts interchanges I-137 and I-109.

Accordingly, we conclude as we did for I-109 that since the denominator does not contain n, whereas the numerator does, and only in the term containing the parameter, then there can be no cancellation of factors holding for all n in the recursion product for the descending series solutions, Case II.

## SECTION 21

## NON-TELESCOPING OF THE ASCENDING SERIES SOLUTIONS, CASE II

Using I-19 and I-54 in I-33, we have the recursion product for the ascending series solutions

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_2(k+\phi j)^2 + (B_2-A_2)(k+\phi j) + L_2}{\prod_{\phi=1}^{\theta} A_1(k+\phi j)^2 + (B_1-A_1)(k+\phi j) - A_1k^2 - (B_1-A_1)k} \, c^n_k$$

$$= (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_2j^2\phi^2 + 2A_2j\phi k + A_2k^2 + j(B_2-A_2)\phi + (B_2-A_2)k + L_2}{\prod_{\phi=1}^{\theta} A_1j^2\phi^2 + 2A_1j\phi k + j(B_1-A_1)\phi} \, c^n_k$$

$$= \left(\frac{-1}{j}\right)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_2j^2\phi^2 + 2A_2j\phi k + A_2k^2 + j(B_2-A_2)\phi + (B_2-A_2)k + L_2}{\prod_{\phi=1}^{\theta} \phi\,(A_1j\phi + 2A_1k + B_1-A_1)} \, c^n_k \qquad \text{I-138}$$

Put $(\phi+1)$ for $\phi$ in the denominator and obtain

$$c^n_{k+\theta j} = \left(\frac{-1}{j}\right)^{\theta} \frac{\prod\limits_{\phi=0}^{\theta-1} A_2 j^2 \phi^2 + 2A_2 j\phi k + A_2 k^2 + j(B_2-A_2)\phi + (B_2-A_2)k + L_2}{\prod\limits_{\phi=1-i}^{\theta-i} (\phi+i)\left[A_1 j\phi + 2A_1 k + (A_1 ji + B_1 - A_1)\right]} c^n_k \qquad \text{I-139}$$

The situation here is converted into that of I-111, the recursion product for the descending series solutions, Case I, by an interchange of subscripts. Hence, the methods applied to the study of that case, and the results obtained there, as summarized following I-114 and in the chart I-122, can be transferred to this case of I-139, the recursion product for the ascending series solutions, Case II, by an interchange of subscripts in the work following I-115.

# PART II

# NON-CLASSICAL ORTHOGONAL POLYNOMIAL SOLUTIONS with ALGEBRAIC WEIGHT FUNCTION

## SECTION I

## ORDER AND SUBDEGREE OF THE POLYNOMIAL SOLUTIONS

In this second part of the dissertation we consider the polynomial solutions which obtain when the bracket in I-52 is zero; that is, when

$$A_1(B_2-A_2)^2 - A_2K_2^* (B_1-A_1)^2 = 0 \qquad \text{II-1}$$

Multiplying both sides by $A_2K_2^*$ and transposing gives

$$A_2^2 K_2^{*2}(B_1-A_1)^2 = A_1A_2K_2^* (B_2-A_2)^2$$

Taking the square root of both sides gives

$$A_2K_2^* (B_1-A_1) = \pm \sqrt{A_1A_2K_2^*} (B_2-A_2) \qquad \text{II-2}$$

Using this in I-51 gives

$$k(n) = \frac{-(B_2-A_2) \pm \sqrt{4A_1A_2K_2^* G^2n^2 \pm 4\sqrt{A_1A_2K_2^*} (B_2-A_2)Gn + (B_2-A_2)^2}}{2A_2} \qquad \text{II-3}$$

From this we have

$$k(n) = \frac{-(B_2-A_2) \pm \left[ 2\sqrt{A_1A_2K_2^*}\ Gn \pm (B_2-A_2) \right]}{2A_2} \qquad \text{II-4}$$

This gives rise to four possible results for $k(n)$. Since for polynomials I-48 requires $G > 0$, then for $k(n) > 0$, the conditions indicated must hold:

$$k_1(n) = \frac{\sqrt{A_1A_2K_2^*}}{A_2}\ Gn \quad , \ A_2 > 0, \quad A_1A_2K_2^* > 0 \qquad \text{II-5}$$

$$k_2(n) = -\frac{\sqrt{A_1A_2K_2^*}}{A_2}\ Gn \quad , \ A_2 < 0, \quad A_1A_2K_2^* > 0 \qquad \text{II-6}$$

$$k_3(n) = \frac{-(B_2-A_2) + \sqrt{A_1A_2K_2^*}\ Gn}{A_2}, \ A_2 > 0, \quad A_1A_2K_2^* > 0, \quad B_2 < A_2 \qquad \text{II-7}$$

$$k_4(n) = \frac{-(B_2-A_2) - \sqrt{A_1A_2K_2^*}\ Gn}{A_2}, \ A_2 < 0, \quad A_1A_2K_2^* > 0, \quad B_2 > A_2 \qquad \text{II-8}$$

Let

$$u = \frac{B_1}{A_1}, \qquad v = \frac{B_2}{A_2} \qquad \text{II-9}$$

It will be shown later following II-45 that the existence of a real interval of orthogonality and the nonvanishing of the integrated square of $y_0$ together will demand that

$$v > u > 1 \qquad \text{II-10}$$

Accordingly, if $A_2 > 0$, then by II-9, we have $B_2 > A_2$, contrary to the requirement in II-7 for the possibility $k_3(n)$.

Again, by II-10 and II-9, if $A_2 < 0$, then $B_2 < A_2$, contrary to the conditions required in II-8 for the possibility $k_4(r)$.

Therefore we can have only the possibilities $k_1(r)$ and $k_2(r)$ of II-5 and II-6, respectively. Since these considered together give $k(r) > 0$, whether $A_2 \gtrless 0$, we combine the two results by introducing the $A_2$ into the radical to obtain with the aid of I-50

$$k(r) = \sqrt{\frac{A_1K_2}{A_2K_1}}\, Gn \qquad \text{II-11}$$

From this we see that k(n) will be an integer for all n, so that we may have polynomial solutions, provided

$$N = \sqrt{\frac{A_1K_2}{A_2K_1}}\, G \qquad \text{II-12}$$

is an integer $>1$. Then we may write

$$k(n) = Nn \qquad \text{II-13}$$

Now using I-50 in II-1 and multiplying by $K_1$ gives

$$A_1K_1(B_2-A_2)^2 = A_2K_2(B_1-A_1)^2 \qquad \text{II-14}$$

Using II-9 in this, we have,

$$A_1K_1(vA_2-A_2)^2 = A_2K_2(uA_1-A_1)^2$$

$$A_1A_2^2\, K_1(v-1)^2 = A_2A_1^2\, K_2(u-1)^2$$

Then for $A_1, A_2 \neq 0$, we have as our requirement for rational k(n)

$$A_2 K_1 (v-1)^2 = A_1 K_2 (u-1)^2 \qquad \text{II-15}$$

Rearrangement of this gives

$$\frac{A_1 K_2}{A_2 K_1} = \frac{(v-1)^2}{(u-1)^2}$$

Taking square roots gives

$$\sqrt{\frac{A_1 K_2}{A_2 K_1}} = \frac{v-1}{u-1} \qquad \text{II-16}$$

where the positive root is taken on the right because of II-10.

Now using II-9, we have

$$\frac{v-1}{u-1} = \frac{\frac{B_2}{A_2} - 1}{\frac{B_1}{A_1} - 1} = \frac{A_1(B_2 - A_2)}{A_2(B_1 - A_1)} \qquad \text{II-17}$$

Combining II-12, II-16, and II-17, we have, using II-10

$$\frac{N}{G} = \sqrt{\frac{A_1K_2}{A_2K_1}} = \frac{A_1(B_2-A_2)}{A_2(B_1-A_1)} = \frac{v-1}{u-1} > 0 \qquad \text{II-18}$$

## SECTION II

## COMPATIBILITY OF ORDER, SUBDEGREE, JUMP, AND PARAMETER

For compatibility of the order k(n), subdegree h(n), and jump j of the descending polynomial solutions, we must have for all n

$$\frac{k(n) - h(n)}{j} = \text{an integer} < 0 \qquad \text{II-19}$$

Using I-48 and II-15, this requirement becomes

$$\frac{N-G}{j}\, n = \text{an integer} < 0 \qquad \text{II-20}$$

which means that we must have for descending polynomial solutions

$$\frac{N-G}{j} = -b \qquad \text{II-21}$$

where b is a positive integer. Using II-18 in this, we have

$$\frac{1}{j}\left[\frac{A_1(B_2-A_2)}{A_2(B_1-A_1)}G - G\right] = -b$$

The requirement may then be written

$$\frac{G}{j}\left[\frac{A_1(B_2-A_2)}{A_2(B_1-A_1)} - 1\right] = -b \qquad \text{II-22}$$

Combining the terms in the bracket, we have

$$\frac{G}{j}\left[\frac{A_1(B_2-A_2) - A_2(B_1-A_1)}{A_2(B_1-A_1)}\right] = -b \qquad \text{II-23}$$

Expanding the terms in the bracket, we have

$$\frac{G}{j}\left[\frac{A_1B_2 - A_2B_1}{A_2(B_1 - A_1)}\right] = -b$$

We may write this in determinant form as

$$\frac{G \begin{vmatrix} B_1 & B_2 \\ A_1 & A_2 \end{vmatrix}}{A_2(B_1 - A_1)j} = b \qquad \text{II-24}$$

From this we see that G must contain

$$\frac{A_2(B_1-A_1)j}{J} \qquad \text{II-25}$$

where

$$J = \text{g.c.d. of } A_2(B_1-A_1)j \text{ and } \begin{vmatrix} B_1 & B_2 \\ A_1 & A_2 \end{vmatrix} \qquad \text{II-26}$$

or, by II-22, if $|j| = 1$, then

$$J = \text{g.c.d. of } A_2(B_1-A_1) \text{ and } A_1(B_2-A_2)$$

Now the closure equation solved for $L(n)$ is

$$L(n) = \frac{-A_1 G^2 n^2 - (B_1 - A_1) G n}{K_1} \tag{I-49'}$$

From this we see that for the parameter $L(n)$ to be an integer for all n, then the coefficients of both powers of n must be integers. Hence $G^2$ must contain $\frac{K_1}{E}$, where

$$E = \text{g.c.d. of } A_1 \text{ and } K_1 \tag{II-27}$$

That is, G must contain

$$\sqrt{\left|\frac{K_1}{E}\right|} \tag{II-28}$$

Also, G must contain

$$\frac{K_1}{F} \tag{II-29}$$

where

$$F = \text{g.c.d. of } (B_1 - A_1) \text{ and } K_1 \tag{II-30}$$

Accordingly,

$$G^* = \left| A_2 K_1 (B_1 - A_1) j \right| \qquad \text{II-31}$$

will certainly contain II-25, II-28, and II-29, and with a surplus, as indicated by the brackets in the following equations:

$$G^* = \left| \frac{A_2(B_1 - A_1)j}{J} \, [JK_1] \right|$$

$$= \left| \sqrt{\left|\frac{K_1}{E}\right|} \, \left[A_2(B_1 - A_1)j \sqrt{|K_1 E|}\right] \right|$$

$$= \left| \frac{K_1}{F} \, \left[A_2(B_1 - A_1)Fj\right] \right|$$

Hence, the least value of G which will make II-24 and I-49' integers is

$$G = \frac{G^*}{Q} \qquad \text{II-32}$$

where

$$Q = \text{g.c.d. of } \; JK_1, \; A_2(B_1 - A_1)j\sqrt{|K_1 E|}, \text{ and } \; A_2(B_1 - A_1)Fj \qquad \text{II-33}$$

That is,

$$G = \left| \frac{K_1 A_2 (B_1 - A_1) J}{Q} \right| \qquad \text{II-34}$$

where Q, J, E, F, are given by II-33, II-26, II-27, and II-30, respectively.

Thus we have shown that is always possible to find polynomial solutions of I-1 if II-1 and II-10 hold.

To obtain the values of N and G it may, however, be more expedient to use II-16 to get $\frac{N}{G}$, and then to proceed by inspection to find the precise value of G required, as will be illustrated by the numerical examples of section 13, rather than to follow the formal procedure outlined above.

## SECTION 3

## THE INTEGRATING FACTOR

We have from preceding considerations

$$wp = e^{\int \frac{q}{p} dx} \qquad \text{I-9}'$$

Using I-2 and I-3 in this, we have

$$wp = e^{\int \frac{B_1 x^{r+1} + B_2 x^{s+1}}{A_1 x^{r+2} + A_2 x^{s+2}}\, dx} \qquad \text{II-35}$$

Using II-9 in this we have

$$wp = e^{\int \frac{uA_1 x^{r+1} + vA_2 x^{s+1}}{A_1 x^{r+2} + A_2 x^{s+2}}\, dx}$$

Adding and subtracting

$$\frac{uA_2 x^{s+1}}{A_1 x^{r+2} + A_2 x^{s+2}}$$

in the integrand, we have

$$wp = e^{\int \left[\frac{uA_1 x^{r+1} + uA_2 x^{s+1}}{A_1 x^{r+2} + A_2 x^{s+2}} + \frac{(v-u)A_2 x^{s+1}}{A_1 x^{r+2} + A_2 x^{s+2}}\right] dx}$$

Dividing out the $A_1x^{r+1} + A_2x^{s+1}$ in the first term of the integrand and $x^{r+2}$ in the second, we have

$$wp = e^{u \int \frac{dx}{x} + (v-u) \int \frac{A_2x^{s-r-1}}{A_1 + A_2x^{s-r}} dx}$$

Putting $r-s = j < 0$, as was done for the descending series solutions in I-27, we have

$$wp = e^{u \int \frac{dx}{x} + \frac{v-u}{-j} \int \frac{-jA_2x^{-j-1}}{A_1 + A_2x^{-j}} dx}$$

Integrating, we have

$$wp = e^{u \ln x + \frac{u-v}{j} \ln (A_1 + A_2x^{-j})}$$

$$= e^{\ln x^u + \ln (A_1 + A_2x^{-j})^{\frac{u-v}{j}}}$$

$$wp = e^{\ln x^{u}(A_1 + A_2 x^{-j})^{\frac{u-v}{j}}}$$

We may write this in the following three forms:

$$wp = x^{u}(A_1 + A_2 x^{-j})^{\frac{u-v}{j}} \qquad \text{II-36}$$

$$= \frac{(A_1 + A_2 x^{-j})^{\frac{u-v}{j}}}{x^{-u}} \qquad \text{II-37}$$

$$= \frac{x^{u}}{(A_1 + A_2 x^{-j})^{\frac{v-u}{j}}} \qquad \text{II-38}$$

Dividing II-36 by p and using I-2, we have the integrating factor

$$w = \frac{x^{u}(A_1 + A_2 x^{-j})^{\frac{u-v}{j}}}{A_1 x^{r+2} + A_2 x^{s+2}}$$

Again using $r-s = j$, we have the integrating factor in the form

$$w = x^{u-r-2}\left(A_1 + A_2 x^{-j}\right)^{\frac{u-v}{j} - 1} \qquad \text{II-39}$$

## SECTION 4

## THE INTERVAL OF ORTHOGONALITY

From II-36, we will have real roots of $wp$ at

$$x = 0,\ (\pm 1)^{-j+1} \sqrt[-j]{-\frac{A_1}{A_2}}$$

if $v > u > 0$, and if $\frac{A_1}{A_2} < 0$ for $j$ an even negative integer. II-40

From II-37, $wp$ will have real roots at

$$x = \infty,\ (\pm 1)^{-j+1} \sqrt[-j]{-\frac{A_1}{A_2}}$$

if $u < v < 0$, and if $\frac{A_1}{A_2} < 0$ for $j$ an even negative integer. II-41

From II-36, wp will have real roots at

$$x = 0, \pm \infty$$

$$\text{if } v < o < u \qquad \text{II-42}$$

The possibilities with infinite root are, however, untenable. For, since $y_n$ are polynomials of order $k(n)$, where $k(n)$ is an increasing function of n, whereas u and v are fixed constants, then sooner or later, as n increases, the order of the bracket in I-7, namely

$$wp(y_m' y_n - y_m y_n')$$

will become $> 0$, and hence become infinite for $x = \pm \infty$, instead of vanishing, as required by I-7 for the establishment of the orthogonality relation I-11.

Accordingly, we can retain only the possibility of II-40. This establishes the inequality

$$v > u > o \qquad \text{II-43}$$

But if we are to be able to express the integrated square by formula, it will be shown later following II-117 that we must also have

$$v-u-1 = 0 \qquad \text{II-44}$$

## SECTION 5

## THE WEIGHT FUNCTION

Using II-39 in I-24, we have

$$W = x^{u-r-2} (A_1 + A_2x^{-j})^{\frac{u-v}{j} - 1} (K_1x^r + K_2x^s)$$

Combining with the first factor an $x^r$ taken from the last factor, and from I-27 using $r-s = j$, we have as the weight function for the descending series solutions

$$W = x^{u-2} (A_1 + A_2x^{-j})^{\frac{u-v}{j} - 1} (K_1 + K_2x^{-j}) \qquad \text{II-45}$$

Now from II-13 and I-48, we have

$$k(o) = h(o) = 0$$

so that $y_0$ is a constant. Accordingly, using II-40 and II-45 in I-25, with $m = n = o$, we have for the integrated square of $y_0$ the integral

$$y_0^2 \int_0^{(\pm 1)^{-j+1} \sqrt[-j]{-\frac{A_1}{A_2}}} x^{u-2} \left(A_1 + A_2 x^{-j}\right)^{\frac{u-v}{j} - 1} \left(K_1 + K_2 x^{-j}\right) dx \qquad \text{II-46}$$

The expansion of the binomials in the integrand results in a series beginning with a constant term. The product of this constant and the factor $x^{u-2}$ must be x to a power greater than (-1), if the integral is to converge for the lower limit. That is, we must have

$$u-2 > -1$$

which means that

$$u > 1 \qquad \text{II-47}$$

Combining this last result II-47 with II-43 gives us

$$v > u > 1 \qquad \text{II-10'}$$

## SECTION 6

## COMPATIBILITY OF SIGNS OF THE COEFFICIENTS IN THE DIFFERENTIAL EQUATION

The indicial equation I-45 for the descending series solutions solved for $L(n)$ is

$$L(n) = \frac{-A_2 k^2(n) - (B_2 - A_2)\, k(n)}{K_2} \qquad \text{II-48}$$

The closure equation I-47 for the descending series solutions solved for $L(n)$ is

$$L(n) = \frac{-A_1 h^2(n) - (B_1 - A_1)\, h(n)}{K_1} \qquad \text{II-49}$$

Since for polynomials we must have $k(n) \geqslant h(n) \geqslant 0$ for all $n$, then from these last two equations and the fact that $A_1 > 0$ by I-1 ff., we conclude that

$$\text{For } L(n) \geqslant 0, \text{ then } \overset{+}{A_1}, \overset{-}{K_1} \text{ have opposite signs}$$
$$\text{and } A_2, K_2 \text{ have opposite signs} \qquad \text{II-50}$$

For $L(n) \leqslant 0$, then $\overset{+}{A}_1$, $\overset{+}{K}_1$ have the same signs
and $A_2$, $K_2$ have the same signs. II-51

From II-43 and II-9 we conclude that

For $u > o$, then $\overset{+}{A}_1$, $\overset{+}{B}_1$ have the same signs.

For $v > o$, then $A_2$, $B_2$ have the same signs. II-52

From II-40 we conclude that

For real interval, positive if $j$ is odd,

then $\overset{+}{A}_1$, $\overset{-}{A}_2$ have opposite signs. II-53

For real negative interval if $j$ is odd,

then $\overset{+}{A}_1$, $\overset{+}{A}_2$ have the same signs. II-54

Accordingly, from II-50, II-52, II-53 we se that for real interval, positive if $j$ is odd, and, $L(n) \geqslant 0$, then we must have

$$\overset{+}{A}_1,\ \overset{-}{A}_2,\ \overset{+}{B}_1,\ \overset{-}{B}_2,\ \overset{-}{K}_1,\ \overset{+}{K}_2 \qquad \text{II-55}$$

From II-51, II-52, II-53 we see that for real interval, positive if $j$ is odd, and $L(r) \leqslant 0$, then we must have

$$\overset{+}{A}_1,\ \overset{-}{A}_2,\ \overset{+}{B}_1,\ \overset{-}{B}_2,\ \overset{+}{K}_1,\ \overset{-}{K}_2 \qquad \text{II-56}$$

From II-50, II-52, II-54 we see that for real negative interval if $j$ is odd, and $L(r) \geqslant 0$, then we must have

$$\overset{+}{A}_1,\ \overset{+}{A}_2,\ \overset{+}{B}_1,\ \overset{+}{B}_2,\ \overset{-}{K}_1,\ \overset{-}{K}_2 \qquad \text{II-57}$$

From II-51, II-52, II-54 we see that for real negative interval if $j$ is odd, and $L(r) \leqslant 0$, then we must have

$$\overset{+}{A}_1,\ \overset{+}{A}_2,\ \overset{+}{B}_1,\ \overset{+}{B}_2,\ \overset{+}{K}_1,\ \overset{+}{K}_2 \qquad \text{II-58}$$

We recall that II-11 imposed the condition that $A_1$, $A_2$, $K_1$, $K_2$ be all positive, or else contain two negative signs. We observe that the results II-55, II-56, II-57, II-58 are all in accord with this requirement.

Following II-98 it will be shown that for $u$ a fraction, then II-57 and II-58 cannot obtain.

## SECTION 7

## RECURSION PRODUCT FOR THE DESCENDING POLYNOMIAL SOLUTIONS

Using I-22 in I-30, we have as the recursion product for the descending series solutions

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod\limits_{\phi=0}^{\theta-1} A_1(k+\phi j)^2 + (B_1-A_1)(k+\phi j) + K_1 \; L(n)}{\prod\limits_{\phi=1}^{\theta} A_2(k+\phi j)^2 + (B_2-A_2)(k+\phi j) + K_2 \; L(n)} c^n_k \qquad \text{II-59}$$

Now let

$$K_1^* = \frac{K_1}{K_2} \qquad \text{II-60}$$

Using this and II-48 in II-59 and expanding, we have

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod\limits_{\phi=0}^{\theta-1} A_1 j^2 \phi^2 + 2A_1 j\phi k + A_1 k^2 + j(B_1 - A_1)\phi + (B_1 - A_1)k - K_1^* A_2 k^2 - K_1^*(B_2 - A_2)k}{\prod\limits_{\phi=1}^{\theta} A_2 j^2 \phi^2 + 2A_2 j\phi k + A_2 k^2 + j(B_2 - A_2)\phi + (B_2 - A_2)k - A_2 k^2 - (B_2 - A_2)k} c^n_k$$

$$c^{n}_{k+\theta j} = \left(\frac{-1}{j}\right)^{\theta} \frac{\displaystyle\prod_{\phi=0}^{\theta-1} A_1 j^2 \phi^2 + 2A_1 j\phi k + (A_1 - A_2 K_1^*) k^2 + j(B_1 - A_1)\phi + \left[(B_1 - A_1) - K_1^*(B_2 - A_2)\right] k}{\displaystyle\prod_{\phi=1}^{\theta} \phi\left[A_2 j\phi + 2A_2 k + (B_2 - A_2)\right]} c^{n}_{k} \qquad \text{II-61}$$

Squaring II-18 gives

$$\frac{G^2}{N^2} = \frac{A_2 K_1}{A_1 K_2}$$

Using II-60 in this gives

$$A_2 K_1^* N^2 = A_1 G^2 \qquad \text{II-62}$$

By II-9 we have

$$B_1 - A_1 = A_1(u-1) \qquad \text{II-63}$$

and

$$B_2 - A_2 = A_2(v-1) \qquad \text{II-64}$$

Using II-18, II-63, II-64, and II-60, we have

$$\left[(B_1-A_1) - K_1^*(B_2-A_2)\right] N = \left[A_1(u-1) - A_1\left(\frac{A_2K_1}{A_1K_2}\right)(v-1)\right] N$$

$$= A_1\left[(u-1) - \frac{G(u-1)}{N(v-1)}(v-1)\right] N$$

$$= A_1(N-G)(u-1) \qquad \text{II-65}$$

Using II-13, II-62, II-63, II-65 in the numerator of II-61, and II-13, II-64 in the denominator, we have, after factoring $A_1$ and $A_2$ out of the numerator and denominator, respectively,

$$c^r_{k+\theta j} = \left(-\frac{A_1}{A_2 j}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} j^2\phi^2 + 2jN\phi n + (N^2-G^2)n^2 + j(u-1)\phi + (N-G)(u-1)n}{\prod_{\phi=1}^{\theta} \phi\left[j\phi + 2Nn + (v-1)\right]} c^n_k \qquad \text{II-66}$$

Now in the discussion of closure of solutions following I-37, we let θ(r) to the value of θ for which we obtained the last non-vanishing coefficient $c^n_{k+\theta j}$ in the descending series solution, and showed that then the numerator of the recursion product I-30 had to be zero for $\phi_{max} = \Theta(n)$.

Accordingly, the numerator of the recursion product II-66 must be zero for $\theta$ having the value $\Theta(n)$, which we have previously shown is related to the order k(n) and subdegree h(n) by the equation

$$h(n) = k(n) + \Theta(n)j \qquad \text{I-39'}$$

from which we have

$$\Theta(n) = \frac{h(n) - k(n)}{j}$$

Using I-48 and II-13 in this, we have

$$\Theta(n) = \frac{G-N}{j}\, n \qquad \text{II-67}$$

By II-21 this becomes

$$\Theta(n) = bn \qquad \text{II-68}$$

Since the numerator of the recursion product II-66 is zero for $\theta = \theta(n)$, then

$$\phi - \theta(n)$$

must be a factor of it. Using II-67, we have this factor as

$$\phi - \frac{G-N}{j}\,n$$

Multiplying by the constant j, we take this factor in the form

$$j\phi + (N-G)n \qquad\qquad \text{II-69}$$

To obtain the other factor of the numerator of II-66, we divide this numerator by the factor of II-69, as follows:

$$\begin{array}{r|l}
 & j\phi + (N+G)n + (u-1) \\ \hline
j\phi + (N-G)n & j^2\phi^2 + 2jN\phi n + (N^2-G^2)n^2 + j(u-1)\phi + (N-G)(u-1)n \\
 & \underline{j^2\phi^2 + j(N-G)\phi n} \\
 & \quad j(N+G)\phi n \\
 & \quad \underline{j(N+G)\phi n + (N^2-G^2)n^2} \\
 & \qquad\qquad \underline{j(u-1)\phi + (N-G)(u-1)n} \\
 & \qquad\qquad\qquad 0
\end{array} \qquad \text{II-70}$$

Using II-69 and II-70 in II-66, and this result, together with II-13 and II-68 in I-26, we have the Non-Classical Orthogonal Polynomial Solutions

$$y_n = c_k^n \sum_{\theta=0}^{bn} \left(-\frac{A_1}{A_2 j}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} \left[(N-G)n + j\phi\right]\left[(N+G)n + (u-1) + j\phi\right]}{\theta! \prod_{\phi=1}^{\theta} \left[2Nn + (v-1) + j\phi\right]} x^{Nn+\theta j} \qquad \text{II-71}$$

For the particular value $j = -1$, this becomes

$$y_n = c_k^n \sum_{\theta=0}^{bn} \left(\frac{A_1}{A_2}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} \left[(N-G)n - \phi\right]\left[(N+G)n + (u-1) - \phi\right]}{\theta! \prod_{\phi=1}^{\theta} \left[2Nn + (v-1) - \phi\right]} x^{Nn-\theta} \qquad \text{II-72}$$

## SECTION 8

## DERIVATIVE DEFINITION OF THE POLYNOMIAL SOLUTIONS

We now consider the functions $Y_n$ given by the derivative definition

$$Y_n = D^R\left[x^S (A_2 x^{-j} + A_1)^T\right] \qquad \text{II-73}$$

where D is the derivative operator $\frac{d}{dx}$; R,S,T constants; T an integer; $R < S$.

If we impose the condition that R be an integer, then our derivatives are taken in the ordinary sense, namely

$$D^R x^\phi = \phi[\phi-1] \ldots [\phi-(R-1)]\, x^{\phi-R}$$

which, if $\phi$ is an integer, we may write as

$$D^R x^\phi = \frac{\phi!}{(\phi-R)!}\, x^{\phi-R}$$

or, if $\phi$ is not an integer, as

$$D^R x^\phi = \frac{\Gamma(\phi+1)}{\Gamma(\phi-R+1)}\, x^{\phi-R} \qquad \text{II-74}$$

which, by the deffinition of the $\Gamma$ function, would of course include the case of $\phi$ an integer.

However, we may admit formal derivatives of fractional order, defined by this same relation II-74, R being taken as a fraction; $\phi$ an integer or fraction as before.

Formal derivatives of fractional order for the general function $f(x)$ are defined by the formula (3, Vol. II, p. 339)

$$D^{R} f(x) = D^{m} D^{-\theta} f(x) = D^{m} \frac{1}{\Gamma(\theta)} \int_{0}^{x} (x-t)^{\theta-1} f(t)\, dt \qquad \text{II-75}$$

where $R = m - \theta, \quad 0 < \theta < 1$

and $m =$ the smallest integer $> R$;

that is, $= [R] + 1$

where $[R] =$ the largest integer in $R$.

$D^{R} f(x)$ exists if $f(x)$ has continuous derivatives up to and including the $m$ th order.

For the power function $f(x) = x^{\phi}$, we have

$$D^{R} x^{\phi} = D^{m} D^{-\theta} x^{\phi} = D^{m} \frac{1}{\Gamma(\theta)} \int_{0}^{x} (x-t)^{\theta-1} t^{\phi}\, dt$$

But the Beta function is given by (12, p. 166, equation 3)

$$B(\theta,\phi+1) = \frac{1}{x^{\theta+(\phi+1)-1}} \int_0^x (x-t)^{\theta-1}\, t^{(\phi+1)-1}\, dt \qquad \text{II-76}$$

provided $\theta > 0$, $\phi > -1$. Hence

$$D^R x^{\phi} = D^m \frac{1}{\Gamma(\theta)} B(\theta,\phi+1)\, x^{\phi+\theta}$$

But (12,p.167, equation 6)

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \text{II-77}$$

Hence

$$D^R x^{\phi} = D^m \frac{1}{\Gamma(\theta)} \frac{\Gamma(\theta)\,\Gamma(\phi+1)}{\Gamma(\phi+\theta+1)} x^{\phi+\theta}$$

$$= \frac{\Gamma(\phi+1)}{\Gamma(\phi+\theta+1)} D^m x^{\phi+\theta}$$

Differentiating by II-74, m being an integer, we have

$$D^{R} x^{\phi} = \frac{\Gamma(\phi+1)}{\Gamma(\phi+\theta+1)} \; \frac{\Gamma(\phi+\theta+1)}{\Gamma(\phi+\theta-m+1)} \, x^{\phi+\theta-m}$$

Since from II-75 $R = m - \theta$, this gives

$$D^{R} x^{\phi} = \frac{\Gamma(\phi+1)}{\Gamma(\phi-R+1)} \, x^{\phi-R} \qquad \text{II-74'}$$

as the definition of formal derivative of fractional order for the power function.

Now expanding the binomial in II-73, we have

$$Y_n = D^{R} x^{S} \sum_{\theta=0}^{T} \binom{T}{\theta} \left(A_2 x^{-j}\right)^{T-\theta} A_1^{\theta}$$

$$= D^{R} \sum_{\theta=0}^{T} A_1^{\theta} A_2^{T-\theta} \binom{T}{\theta} x^{S-(T-\theta)j}$$

$$= A_2^{T} \sum_{\theta=0}^{T} \left(\frac{A_1}{A_2}\right)^{\theta} \binom{T}{\theta} D^{R} x^{S-(T-\theta)j}$$

Carrying out the indicated differentiation using II-74 yields

$$Y_n = A_2^T \sum_{\theta=0}^{T} \left(\frac{A_1}{A_2}\right)^{\theta} \binom{T}{\theta} \frac{\Gamma[S-(T-\theta)j+1]}{\Gamma[S-R-(T-\theta)j+1]} \; x^{S-R-(T-\theta)j} \qquad \text{II-78}$$

Putting $\bar{j} = -j$, expanding and multiplying numerator and denominator by the quantities indicated, we have

$$Y_n = A_2^T \left\{ \frac{\Gamma[S+T\bar{j}+1]}{\Gamma[S-R+T\bar{j}+1]} \, x^{S-R-Tj} + \left(\frac{A_1}{A_2}\right) \frac{T}{1!} \frac{\Gamma[S+(T-1)\bar{j}+1]}{\Gamma[S-R+(T-1)\bar{j}+1]} \right.$$

$$\circ \; \frac{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+T\bar{j}-(\bar{j}-1)]}{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+T\bar{j}-(\bar{j}-1)]}$$

$$\circ \; \frac{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+T\bar{j}-(\bar{j}-1)]}{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+T\bar{j}-(\bar{j}-1)]} \; x^{S-R-(T-1)j}$$

$$+ \left(\frac{A_1}{A_2}\right)^2 \frac{T(T-1)}{2!} \quad \frac{\Gamma[S+(T-2)\bar{j}+1]}{\Gamma[S-R+(T-2)\bar{j}+1]}$$

$$\circ \; \frac{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+T\bar{j}-(\bar{j}-1)]}{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+T\bar{j}-(\bar{j}-1)]}$$

$$\circ \frac{[S+(T-1)\bar{j}]\,[S+(T-1)\bar{j}-1]\ldots[S+(T-1)\bar{j}-(\bar{j}-1)]}{[S+(T-1)\bar{j}]\,[S+(T-1)\bar{j}-1]\ldots[S+(T-1)\bar{j}-(\bar{j}-1)]}$$

$$\circ \frac{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+T\bar{j}-(\bar{j}-1)]}{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+T\bar{j}-(\bar{j}-1)]}$$

$$\circ \frac{[S-R+(T-1)\bar{j}]\,[S-R+(T-1)\bar{j}-1]\circ\circ\circ[S-R+(T-1)\bar{j}-(\bar{j}-1)]}{[S-R+(T-1)\bar{j}]\,[S-R+(T-1)\bar{j}-1]\circ\circ\circ[S-R+(T-1)\bar{j}-(\bar{j}-1)]}$$

$$\cdot\ x^{S-R-(T-2)j} \quad + \ldots$$

$$+ \left(\frac{A_1}{A_2}\right)^T (1) \quad \frac{\Gamma(S+1)}{\Gamma(S-R+1)} \circ \frac{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+1]}{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+1]}$$

$$\left.\circ \frac{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+1]}{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+1]}\ x^{S-R}\right\} \qquad \text{II-79}$$

$$Y_n = A_2^T \frac{\Gamma(S+T\bar{j}+1)}{\Gamma(S-R+T\bar{j}+1)} \left\{ x^{S-R-Tj} \right.$$

$$+\left(\frac{A_1}{A_2}\right)\binom{T}{1}\frac{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+T\bar{j}-(\bar{j}-1)]}{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+T\bar{j}-(\bar{j}-1)]}\,x^{S-R-(T-1)j}$$

$$+\left(\frac{A_1}{A_2}\right)^2\binom{T}{2}\frac{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+(T-1)\bar{j}-(\bar{j}-1)]}{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+(T-1)\bar{j}-(\bar{j}-1)]}\,x^{S-R-(T-2)j}$$

$$\left.+\ \ldots\ +\left(\frac{A_1}{A_2}\right)^T\binom{T}{T}\frac{[S-R+T\bar{j}]\,[S-R+T\bar{j}-1]\ldots[S-R+1]}{[S+T\bar{j}]\,[S+T\bar{j}-1]\ldots[S+1]}\,x^{S-R}\right\} \qquad \text{II-79}$$

$$Y_n = A_2^T\,\frac{\Gamma(S+T\bar{j}+1)}{\Gamma(S-R+T\bar{j}+1)}\sum_{\theta=0}^{T}\left(\frac{A_1}{A_2}\right)^{\theta}\binom{T}{\theta}$$

$$\cdot\ \frac{\prod_{\phi=0}^{\theta-1}\ \prod_{i=0}^{\bar{j}-1}\ S-R+(T-\phi)\bar{j}-i}{\prod_{\phi=1}^{\theta}\ \prod_{i=0}^{\bar{j}-1}\ S+(T-\phi+1)\bar{j}-i}\ x^{S-R-(T-\theta)j} \qquad \text{II-80}$$

wherein as following I-30 we define

$$\prod_{0}^{-1} = \prod_{1}^{0} = 1$$

For the particular value $j = -1$, we have

$$Y_n = A_2^{T} \frac{\Gamma(S+T+1)}{\Gamma(S-R+T+1)} \sum_{\theta=0}^{T} \left(\frac{A_1}{A_2}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} (T-\phi)(S-R+T-\phi)}{\theta! \prod_{\phi=1}^{\theta} (S+T+1-\phi)} \, x^{S-R+T-\theta} \qquad \text{II-81}$$

Comparing II-81 with II-72, we see that in order to be able to identify the functions $Y_n$ of II-80, which we obtained from definition II-73, with the solutions $y_n$ of II-71, we must take $j = -1$. Then in II-72 and II-81 we make the following identifications:

$$\begin{array}{lcl|c|c|c|r}
T & = & (N-G)\,n & & -1 & & \text{II-82} \\
S + T + 1 & = & (2N)\,n + (v-1) & +1 & +1 & & \text{II-83} \\
S - R + T & = & (N+G)\,n + (u-1) & -1 & & & \text{II-84} \\
\hline
R + 1 & = & (N-G)\,n + (v-u) & & & & \\
R & = & (N-G)\,n + (v-u-1) & & & -1 & \text{II-85} \\
\hline
S + 1 & = & (N+G)\,n + (v-1) & & & & \text{II-86} \\
S & = & (N+G)\,n + (v-2) & & & +1 & \text{II-87} \\
\hline
S - R & = & (2G)\,n + (u-1) & & & & \text{II-88}
\end{array}$$

$$c_k^n = A_2^T \frac{\Gamma(S+T+1)}{\Gamma(S-R+T+1)} \qquad \text{II-89}$$

By II-82, II-83, II-84, II-13 this becomes

$$c_{Nn}^n = A_2^{(N-G)n} \frac{\Gamma\left[2Nn + (v-1)\right]}{\Gamma\left[(N+G)n + u\right]} \qquad \text{II-90}$$

From the last term in II-79 we have

$$c_h^n = A_1^T \frac{\Gamma(S+1)}{\Gamma(S-R+1)} \qquad \text{II-91}$$

By II-82, II-86, II-88, I-48 this becomes

$$c_{Gn}^n = A_1^{(N-G)n} \frac{\Gamma\left[(N+G)n + (v-1)\right]}{\Gamma\left[2Gn + u\right]} \qquad \text{II-92}$$

With the preceding identification of the constants in II-72 and II-81, we now have

$$y_n = x^H Y_n \qquad \text{II-93}$$

Equating the orders of both members of this equation gives us

$$k(n) = H + (S-R+T)$$

By II-13 and II-84 this becomes

$$Nn = H + (N+G)n + (u-1)$$

so that H is given by

$$H = -Gn-u+1 \qquad \text{II-94}$$

This same result is also obtainable by equating, with the aid of the last term in II-79, the subdegrees of both members of II-93,

$$h(n) = H + (S-R)$$

By I-48 and II-88 this becomes

$$Gr = H + 2Gn + (u-1)$$

so that H is given by

$$H = -Gn-u+1 \qquad \text{II-94'}$$

Using II-85, II-87, II-82 in II-73, and the result, together with II-94, in II-93, we have as the Derivative Definition of our Non-Classical Orthogonal Polynomial Solutions

$$y_n = x^{-Gn-u+1} D^{(N-G)n+(v-u-1)}\left[x^{(N+G)n+(v-2)}(A_2x + A_1)^{(N-G)n}\right] \qquad \text{II-95}$$

If R is an integer, then II-85 requires that

$$v-u-1 = \text{an integer}$$

But if R is a fraction, then v-u-1 is not necessarily an integer, It is only necessary that

$$v > u > 0 \qquad \text{II-43'}$$

However, it will be shown later following II-117, as stated previously just before II-44, that if we are to be able to express the integrated square by formula, then we must have the stronger condition

$$v-u-1 = o \qquad \text{II-44'}$$

With the aid of II-44' the Derivative Definition II-95 is written

$$y_n = x^{-Gn-u+1} D^{(N-G)n}\left[x^{(N+G)n+(u-1)} (A_2 x + A_1)^{(N-G)n}\right] \qquad \text{II-96}$$

## SECTION 9

## NON-VANISHING OF THE INTEGRATED SQUARE WHEN $n = 0$

Taking $n = o$, we have from II-96

$$y_o = 1 \qquad \text{II-97}$$

Using this, $j = -1$, and II-44' in II-46, we have for the integrated square of $y_o$

$$I(o) = \int_0^{-\frac{A_1}{A_2}} x^{u-2} \, (K_1 + K_2 x) \, dx \qquad \text{II-98}$$

For this integral to converge at the lower limit, II-47 required only that $u > 1$. Hence $(u-2)$ may be a fraction. In this case, for $I(o)$ to be real, we must have $x > o$. Hence $\frac{A_1}{A_2} < 0$, and since $A_1 > 0$, then we have $\overset{+}{A_1}$, $\overset{-}{A_2}$ as in II-55 and II-56, but II-57 or II-58 cannot obtain.

From II-98 we have

$$I(o) = \int_0^{-\frac{A_1}{A_2}} (K_1 x^{u-2} + K_2 x^{u-1}) \, dx$$

$$= \left[ \frac{K_1 x^{u-1}}{u-1} + \frac{K_2 x^{u}}{u} \right]_0^{-\frac{A_1}{A_2}}$$

$$I(o) = \frac{K_1}{u-1}\left(-\frac{A_1}{A_2}\right)^{u-1} + \frac{K_2}{u}\left(-\frac{A_1}{A_2}\right)^{u}$$

This may be written as

$$I(o) = \frac{1}{u(u-1)}\left(-\frac{A_1}{A_2}\right)^{u-1}\left[\frac{A_2}{A_2}.K_1 u - \frac{A_1}{A_2}K_2(u-1)\right]$$

Again using II-44', we have

$$Io) = \frac{(-1)^{u-1}}{u(u-1)}\ \frac{A_1^{\,u-1}}{A_2^{\,u}}\left[A_2K_1(v-1) - A_1K_2(u-1)\right] \qquad \text{II-99}$$

We have already specified in II-47 that $u > 1$. Hence $u-1 > o$. This means that neither factor in the denominator of II-99 is zero.

We also have from II-15

$$A_2K_1(v-1)^2 - A_1K_2(u-1)^2 = 0 \qquad \text{II-100}$$

and from II-10'(ff.II-47)

$$v > u > 1 \qquad \text{II-10"}$$

Hence the bracket in II-99 cannot vanish.

Therefore, we may conclude that no further conditions than those already stipulated must be met to guarantee the non-vanishing of the integrated square of $y_o$. Since the argument of Part I, Section 2, guaranteed the non-vanishing of the integrated square for $n \neq o$, we have now shown the non-vanishing of the integrated square for all n.

## SECTION 10

## INTEGRATED SQUARE FORMULA

Using II-45 with $j = -1$, and II-95, in I-25 with $m = n$, we have for the integrated square of $y_n$

$$I(n) = \int_o^{-\frac{A_1}{A_2}} x^{u-2} (A_1 + A_2 x)^{v-u-1} (K_1 + K_2 x)\ y_n$$

$$\cdot\ x^{-Gn-u+1}\ D^{(N-G)n+(v-u-1)}\left[x^{(N+G)n+(v-2)}\,(A_2x+A_1)^{(N-G)n}\right]dx \qquad \text{II-101}$$

Let

$$\phi = x^{(N+G)n+(v-2)}\,(A_2x+A_1)^{(N-G)n} \qquad \text{II-102}$$

and denote the derivatives of $\phi$ by superscripts in braces.

Expanding the binomial in II-101 and using II-102, we have, anticipating (v-u-1) an integer,

$$I(n) = \int_0^{-A_1/A_2} x^{-Gn-1} \sum_{\theta=0}^{v-u-1} \binom{v-u-1}{\theta} A_1^{(v-u-1)-\theta}\,(A_2x)^{\theta}$$

$$\cdot\ (K_1+K_2x)\ y_n\ \phi^{\{(N-G)n+(v-u-1)\}}\ dx$$

(For v-u-1 not an integer, the upper limit on $\sum$ would be $\infty$.)

We may write this result as

$$I(n) = A_1^{v-u-1} \sum_{\theta=0}^{v-u-1} \binom{v-u-1}{\theta} \left(\frac{A_2}{A_1}\right)^{\theta} \left[K_1 I_1(n) + K_2 I_2(n)\right] \qquad \text{II-103}$$

where

$$I_1(n) = \int_0^{-A_1/A_2} (x^{-Gn+\theta-1} \, y_n) \, \phi^{\{(N-G)n+(v-u-1)\}} \, dx \qquad \text{II-104}$$

$$I_2(n) = \int_0^{-A_1/A_2} (x^{-Gn+\theta} \, y_n) \, \phi^{\{(N-G)n+(v-u-1)\}} \, dx \qquad \text{II-105}$$

It is convenient to consider $I_2(n)$ first. Let

$$P_{(N-G)n+\theta,\theta} = x^{-Gn+\theta} \, y_n \qquad \text{II-106}$$

where P is a polynomial of order and subdegree as indicated by the respective subscripts. Since $y_n$ is of order $k(n)$ and subdegree $h(n)$, then II-106 becomes

$$P_{(N-G)n+\theta,\theta} = x^{-Gn+\theta} \left[c^{n}_{k(n)} \, x^{k(n)} + \ldots + c^{n}_{h(n)} \, x^{h(n)}\right]$$

Using the values of k(n) and h(n) as given by II-13 and I-48, respectively, we finally obtain

$$P_{(N-G)n+\theta,\theta} = x^{-Gn+\theta}\left(c^{n}_{Nn}\, x^{Nn} + \dots + c^{n}_{Gn}\, x^{Gn}\right)$$

$$= c^{n}_{Nn}\, x^{(N-G)n+\theta} + \dots + c^{n}_{Gn}\, x^{\theta} \qquad \text{II-107}$$

Using II-106 in II-105 gives us

$$I_2(n) = \int_0^{-A_1/A_2} P_{(N-G)n+\theta,\theta}\, \phi^{\left\{(N-G)n+(v-u-1)\right\}}\, dx$$

By parts

$$U = P_{(N-G)n+\theta,\theta} \qquad dV = \phi^{\left\{(N-G)n+(v-u-1)\right\}}\, dx$$

$$dU = P'_{(N-G)n+\theta,\theta}\, dx \qquad V = \phi^{\left\{(N-G)n+(v-u-1)-1\right\}}$$

Then

$$I_2(n) = \left[ P_{(N-G)n+\theta,\theta}\ \phi^{\{(N-G)n+(v-u-1)-1\}} \right]_0^{-A_1/A_2}$$

$$- \int_0^{-A_1/A_2} P'_{(N-G)n+\theta,\theta}\ \phi^{\{(N-G)n+(v-u-1)-1\}}\, dx \qquad \text{II-108}$$

Using the expression for $\phi$ given by II-102 and performing the differentiation indicated in the bracket of II-108, we have

$$\phi^{\{(N-G)n+(v-u-1)-1\}} = D^{v-u-1}\, D^{(N-G)n-1}$$

$$\cdot \left[ x^{(N+G)n+(v-2)} \, (A_2 x + A_1)^{(N-G)n} \right] \qquad \text{II-109}$$

Using Leibniz' Rule to differentiate the product within the brackets, we have for the right member of II-109

$$D^{v-u-1}\left[\sum_{\theta=0}^{(N-G)n-1}\binom{(N-G)n-1}{\theta} D^{(N-G)n-1-\theta}\, x^{(N+G)n+(v-2)} \cdot D^{\theta}\,(A_2x+A_1)^{(N-G)n}\right]$$

Using II-74 this becomes

$$D^{v-u-1}\left\{\sum_{\theta=0}^{(N-G)n-1}\binom{(N-G)n-1}{\theta}\frac{\Gamma[(N+G)n+(v-1)]}{\Gamma(2Gn+v+\theta)}\, x^{2Gn+v-1+\theta} \cdot A_2^{\theta}\,\frac{[(N-G)n]!}{[(N-G)n-\theta]!}\,(A_2x+A_1)^{(N-G)n-\theta}\right\}$$

$$= D^{v-u-1}\left[\left\{\sum_{\theta=0}^{(N-G)n-1}\binom{(N-G)n-1}{\theta} A_2^{\theta}\,\frac{[(N-G)n]!}{[(N-G)n-\theta]!}\,\frac{\Gamma[(N+G)n+(v-1)]}{\Gamma(2Gn+v+\theta)} \cdot x^{2Gn+v-1+\theta}\,(A_2x+A_1)^{(N-G)n-1-\theta}\right\}(A_2x+A_1)\right] \qquad \text{II-110}$$

If instead of using Leibriz' Rule to differentiate the product in the bracket of II-109, we first expand the binomial and then differentiate, we have

$$D^{v-u-1} D^{(N-G)n-1} \left[ x^{(N+G)n+(v-2)} \sum_{\theta=0}^{(N-G)n} \binom{(N-G)n}{\theta} (A_2 x)^{(N-G)n-\theta} A_1^{\theta} \right]$$

$$= D^{v-u-1} D^{(N-G)n-1} \left[ A_2^{(N-G)n} x^{2Nn+v-2} + \ldots + A_1^{(N-G)n} x^{(N+G)n+v-2} \right]$$

$$= D^{v-u-1} \left[ A_2^{(N-G)n} \frac{\Gamma(2Nn+v-1)}{\Gamma\left[(N+G)n+v\right]} x^{(N+G)n+v-1} + \ldots + A_1^{(N-G)n} \frac{\Gamma\left[(N+G)n+(v-1)\right]}{\Gamma(2Gn+v)} x^{2Gn+v-1} \right]$$

Since II-110 shows that the quantity in the bracket contains $(A_2 x + A_1)$ as a factor, we then have

$$D^{v-u-1} \left[ \left\{ A_2^{(N-G)n-1} \frac{\Gamma(2Nn+v-1)}{\Gamma\left[(N+G)n+v\right]} x^{(N+G)n+v-2} + \ldots \right. \right.$$

$$\left. + A_1^{(N-G)n-1} \frac{\Gamma\left[(N+G)n+v-1\right]}{\Gamma(2Gn+v)} x^{2Gn+v-1} \right\} (A_2 x + A_1) \Bigg] \qquad \text{II-111}$$

We note that the summation in the braces of II-110, and hence in the braces of II-111, does not contain the factor $(A_2 x + A_1)$. Accordingly, if this factor is to be preserved in the remaining differentiation in II-110, then we must have

$$v-u-1 < 1$$

But from considerations regarding interval, we have from II-43

$$v-u-1 > -1$$

Combining these two inequalities gives the requirement

$$-1 < v-u-1 < 1 \qquad \text{II-112}$$

Now suppose that $v-u-1$ is negative; that is, suppose

$$-1 < v-u-1 < 0$$

Then using II-75, with $m = 0$, $R = -\theta$, $\theta = -(v-u-1)$, the result II-110 becomes

$$\frac{1}{\Gamma[-(v-u-1)]} \int_0^x (x-t)^{-(v-u-1)-1} \Bigg\{ \sum_{\theta=0}^{(N-G)n-1} \binom{(N-G)n-1}{\theta} A_2^{\theta}$$

$$\cdot \frac{[(N-G)n]!}{[(N-G)n-\theta]!} \frac{\Gamma[(N+G)n+(v-1)]}{\Gamma(2Gn+v+\theta)} t^{2Gn+v-1+\theta}$$

$$\cdot (A_2 t + A_1)^{(N-G)n-1-\theta} \Bigg\} (A_2 t + A_1) \; dt$$

Let

$$Q(\theta) = \frac{1}{\Gamma[-(v-u-1)]} \binom{(N-G)n-1}{\theta} \frac{[(N-G)n]!}{[(N-G)n-\theta]!} \frac{\Gamma[(N+G)n+(v-1)]}{\Gamma(2Gn+v+\theta)} \qquad \text{II-113}$$

Then our previous result becomes

$$\sum_{\theta=0}^{(N-G)n-1} Q(\theta) \; A_2^{\theta} \int_0^x (x-t)^{-(v-u-1)-1}$$

$$\cdot \; t^{2Gn+v-1+\theta} \; (A_2 t + A_1)^{(N-G)n-\theta} \; dt$$

By parts in each such integral of the summation

$$U = t^{2Gn+v-1+\theta} (A_2 t + A_1)^{(N-G)n-\theta}$$

$$dU = \Big\{ (2Gn+v-1+\theta)\, t^{2Gn+v-2+\theta} (A_2 t + A_1)^{(N-G)n-\theta}$$

$$+ A_2 \left[(N-G)n-\theta\right] t^{2Gn+v-1+\theta} (A_2 t + A_1)^{(N-G)n-1-\theta} \Big\}\, dt$$

$$dV = (x-t)^{-(v-u-1)-1}\, dt, \qquad V = \frac{(x-t)^{-(v-u-1)}}{v-u-1}$$

Hence we have

$$\sum_{\theta=0}^{(N-G)n-1} Q(\theta)\, A_2^{\theta} \left\{ \left[ \frac{(x-t)^{-(v-u-1)}}{v-u-1}\, t^{2Gn+v-1+\theta} (A_2 t + A_1)^{(N-G)n-\theta} \right]_0^x \right.$$

$$- \frac{2Gn+v-1+\theta}{v-u-1} \int_0^x (x-t)^{-(v-u-1)}\, t^{2Gn+v-2-\theta} (A_2 t + A_1)^{(N-G)n-\theta}\, dt$$

$$\left. - \frac{A_2 \left[(N-G)n-\theta\right]}{v-u-1} \int_0^x (x-t)^{-(v-u-1)}\, t^{2Gn+v-1+\theta} (A_2 t + A_1)^{(N-G)n-1-\theta}\, dt \right\}$$

Since $-(v-u-1) > 0$, clearly the bracket vanishes, and we have after expanding the binomials in the remaining integrals $(\,|t| \leqslant |x|\,)$

$$\sum_{\theta=0}^{(N-G)n-1} \sum_{r=0}^{\infty} Q(\theta)\ (-1)^{r} \binom{-(v-u-1)}{r} x^{-(v-u-1)-r} A_2^{\theta}$$

$$\cdot \left[ -\frac{2Gn+v-1+\theta}{v-u-1} \int_0^x t^{2Gn+v+r-2+\theta} (A_2 t + A_1)^{(N-G)n-\theta}\, dt \right.$$

$$\left. - \frac{A_2\left[(N-G)n-\theta\right]}{v-u-1} \int_0^x t^{2Gn+v+r-1+\theta} (A_2 t + A_1)^{(N-G)n-1-\theta}\, dt \right]$$

Let

$$Q_1(r,\theta) = Q(\theta)\ (-1)^{r+1} \binom{-(v-u-1)}{r} \frac{2Gn+v-1+\theta}{v-u-1}$$

$$Q_2(r,\theta) = Q(\theta)\ (-1)^{r+1} \binom{-(v-u-1)}{r} \frac{(N-G)n-\theta}{v-u-1} \qquad \text{II-114}$$

$$A_2 t + A_1 = z, \quad t = \frac{z - A_1}{A_2}, \quad dt = \frac{dz}{A_2}, \quad \begin{array}{c|c} t & z \\ \hline 0 & A_1 \\ x & A_2 x + A_1 \end{array} \qquad \text{II-115}$$

Then the previous result becomes

$$\sum_{\theta=0}^{(N-G)n-1} \sum_{r=0}^{\infty} x^{-(v-u-1)-r}$$

$$\cdot \left[ Q_1(r,\theta)\, A_2^{\theta} \int_{A_1}^{A_2x+A_1} \left(\frac{z-A_1}{A_2}\right)^{2Gn+v+r-2+\theta} z^{(N-G)n-\theta} \frac{dz}{A_2}\right.$$

$$\left. + Q_2(r,\theta)\, A_2^{\theta+1} \int_{A_1}^{A_2x+A_1} \left(\frac{z-A_1}{A_2}\right)^{2Gn+v+r-1+\theta} z^{(N-G)n-1-\theta} \frac{dz}{A_2}\right]$$

Now from II-108, x has the range $(0, -\frac{A_1}{A_2})$. Hence, from II-115, $(A_2x+A_1)$ has the range $(0, \overset{+}{A}_1)$, and therefore z has the range $(0, \overset{+}{A}_1)$. Accordingly, reversing and expanding the binomials in the previous result, we have

$$\sum_{\theta=0}^{(N-G)n-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} x^{-(v-u-1)-r}$$

$$\left[ Q_1(r,\theta)\, (-1)^{2Gn+v+r+s-2+\theta} \binom{2Gn+v+r-2+\theta}{s} \frac{A_1^{2Gn+v+r-s-2+\theta}}{A_2^{2Gn+v+r-1}}\right.$$

$$\cdot \int_{A_1}^{A_2x+A_1} z^{(N-G)n+s-\theta}\, dz$$

$$+\ Q_2(r,\theta)\ (-1)^{2Gn+v+r+s-1+\theta} \binom{2Gn+v+r-1+\theta}{s} \frac{A_1^{\,2Gn+v+r-s-1+\theta}}{A_2^{\,2Gn+v+r-1}}$$

$$\left. \cdot \int_{A_1}^{A_2x+A_1} z^{(N-G)n+s-1-\theta}\, dz \right]$$

Integrating and letting

$$Q_1(r,s,\theta) = Q_1(r,\theta) \binom{2Gn+v+r-2+\theta}{s} \frac{(-1)^{2Gn+v+r+s-2+\theta}}{(N-G)n+s+1-\theta}$$

$$Q_2(r,s,\theta) = Q_2(r,\theta) \binom{2Gn+v+r-1+\theta}{s} \frac{(-1)^{2Gn+v+r+s-1+\theta}}{(N-G)n+s-\theta} \qquad \text{II-116}$$

we have

$$\sum_{\theta=0}^{(N-G)n-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} x^{-(v-u-1)-r} \frac{A_1^{\,2Gn+v+r-s-2+\theta}}{A_2^{\,2Gn+v+r-1}}$$

$$\cdot \left\{ Q_1(r,s,\theta) \left[ (A_2x+A_1)^{(N-G)n+s+1-\theta} - A_1^{(N-G)n+s+1-\theta} \right] \right.$$

$$+ A_1 \cdot Q_2(r,s,\theta)\left[(A_2x + A_1)^{(N-G)n+s-\theta} - A_1^{(N-G)n+s-\theta}\right]\Bigg\}$$

$$= \sum_{\theta=0}^{(N-G)n-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} x^{-(v-u-1)-r} \frac{A_1^{2Gn+v+r-s-2+\theta}}{A_2^{2Gn+v+r-1}}$$

$$\cdot \Bigg\{\left[Q_1(r,s,\theta)\cdot(A_2x + A_1) + Q_2(r,s,\theta)\cdot A_1\right](A_2x + A_1)^{(N-G)n+s-\theta}$$

$$-\left[Q_1(r,s,\theta) + Q_2(r,s,\theta)\right] A_1^{(N-G)n+s+1-\theta}\Bigg\}$$

Thus we see that the factor $(A_2x + A_1)$ is not preserved in the remaining differentiation of II-110 when

$$-1 < v-u-1 < 0$$

We now suppose that $(v-u-1)$ is positive; that is, that

$$0 < v-u-1 < 1$$

Then

$$-1 < (v-u-1) - 1 < 0$$

Using the expression for $\phi$ given by II-102 and performing the differentiation indicated in the bracket of II-108, we have

$$\phi^{\{(N-G)n+(v-u-1)-1\}} = D^{(v-u-1)-1} D^{(N-G)n} \cdot \left[ x^{(N+G)n+(v-2)} (A_2 x + A_1)^{(N-G)n} \right]$$

Using Leibniz' Rule to differentiate the product within the brackets, we have

$$D^{(v-u-1)-1} \left[ \sum_{\theta=0}^{(N-G)n} \binom{(N-G)n}{\theta} D^{(N-G)n-\theta} x^{(N+G)n+(v-2)} \cdot D^{\theta} (A_2 x + A_1)^{(N-G)n} \right]$$

Using II-74 this becomes

$$D^{(v-u-1)-1}\left\{\sum_{\theta=0}^{(N-G)n}\binom{(N-G)n}{\theta}\frac{\Gamma\left[(N+G)n+(v-1)\right]}{\Gamma\left[(2Gn+v-1+\theta)\right]}\,x^{2Gn+v-2+\theta}\right.$$

$$\left.\cdot\,A_2^{\theta}\,\frac{\left[(N-G)n\right]!}{\left[(N-G)n-\theta\right]!}\,(A_2x+A_1)^{(N-G)n-\theta}\right\} \qquad \text{II-117}$$

Let $\sum(x)$ represent the summation in the braces of II-117. Note that $\sum(x)$ does not contain the factor $(A_2x+A_1)$.

Now using II-75 with $m = o$, $R = -\theta$, $\theta = -\left[(v-u-1)-1\right]$, the expression II-117 becomes, with the aid of the above abbreviation

$$\frac{1}{\Gamma\left\{-\left[(v-u-1)-1\right]\right\}}\int_0^x (x-t)^{-(v-u-1)}\sum(t)\,dt$$

This integral is by II-76 a finite series of powers of x, and does not in general contain the factor $(A_2x+A_1)$.

Thus we see that the factor $(A_2x+A_1)$ is not preserved in the remaining differentiation of II-117 when

$$o < v-u-1 < 1$$

Since this was also the case in II-110 when

$$-1 < v-u-1 < 0$$

we must conclude from II-112 that the only tenable value of $(v-u-1)$ is

$$v-u-1 = 0 \qquad \text{II-44}'$$

if the factor $(A_2x + A_1)$ is to be preserved in the bracket of II-108, so that this bracket will vanish at the upper limit $x = -A_1/A_2$, for all $n \neq 0$, since, as seen in II-102, the value $n = 0$ destroys the factor $(A_2x + A_1)$ in $\phi$.

For $v-u-1 = 0$, we see from II-103 that

$$\theta = 0 \qquad \text{II-118}$$

so that II-103 becomes

$$I(n) = K_1I_1(n) + K_2I_2(n) \qquad \text{II-119}$$

and II-108 becomes, with the aid of II-44' and II-111

$$I_2(n) = \left[ P_{(N-G)n,0}\; g_{(N-G)n-1,0}\; x^{2Gn+u} \; (A_2x + A_1) \right]_0^{-A_1/A_2}$$

$$- \int_0^{-A_1/A_2} P'_{(N-G)n,0} \; \phi^{\{(N-G)n-1\}} \, dx \qquad \text{II-120}$$

where $g$ is a polynomial in $x$ of order and subdegree as indicated by the respective subscripts.

For $n > 0$, which we have shown is necessary to preserve the factor $(A_2x + A_1)$ in the bracket of II-120, we see that the bracket in II-120 contains $x$ to at least the power $2+u > 3$, since we have from II-47 that $u > 1$, and we know that $G \geq 1$. Hence the bracket of II-120 vanishes at the lower limit, as well as at the upper limit when $v-u-1 = 0$, as shown above.

If then we integrate by parts in II-105 a total of $(N-G)n$ times, we have, as an extension of II-120,

$$I_2(n) = (-1)^{(N-G)n} \int_0^{A_1/A_2} P^{\{(N-G)n\}}_{(N-G)n,0} \; \phi \, dx$$

since the polynomial in the bracket, as in II-120, will have the same order and subdegree as in II-120, with successive differentiations of P and integrations of $\phi$. By II-107, we have

$$I_2(n) = (-1)^{(N-G)n} \, o^{n}_{Nn} \left[(N-G)n\right]! \int_0^{-A_1/A_2} \phi \, dx$$

Using II-90 and II-102, this becomes, after replacing $(v-1)$ by u from II-44', and after factoring out a $(-A_2)$ from $\phi$,

$$I_2(n) = (-1)^{2(N-G)n} \, A_2^{\,2(N-G)n} \, \frac{\Gamma(2Nn+u)}{\Gamma\left[(N+G)n+u\right]} \left[(N-G)n\right]!$$

$$\cdot \int_0^{-A_1/A_2} x^{(N+G)n+(u-1)} \left[\left(-\frac{A_1}{A_2}\right) - x\right]^{(N-G)n} dx$$

Using II-76 and II-77 this becomes

$$I_2(n) = A_2^{\,2(N-G)n} \, \frac{\Gamma(2Nn+u)}{\Gamma\left[(N+G)n+u\right]} \left[(N-G)n\right]!$$

$$\cdot \left(-\frac{A_1}{A_2}\right)^{2Nn+u} \frac{\Gamma\left[(N+G)n+u\right] \, \Gamma\left[(N-G)n+1\right]}{\Gamma(2Nn+u+1)}$$

$$I_2(n) = (-1)^u \frac{A_1^{2Nn+u}}{A_2^{2Gn+u}} \frac{\left\{[(N-G)n]!\right\}^2}{2Nn+u} \qquad \text{II-121}$$

We now consider $I_1(n)$ as given by II-104. Using $\theta = 0$ from II-118 in this, we let

$$Q_{(N-G)n-1,-1} = x^{-Gn-1} y_n \qquad \text{II-122}$$

where Q is an algebraic function of order and subdegree as indicated by the respective subscripts. Since $y_n$ is of order $k(n)$ and subdegree $h(n)$, then II-122 becomes

$$Q_{(N-G)n-1,-1} = x^{-Gn-1} \left[ c^n_{k(n)} x^{k(n)} + \dots + c^n_{h(n)} x^{h(n)} \right]$$

Using the values of $k(n)$ and $h(n)$ as given by II-13 and I-48, respectively, we finally obtain

$$Q_{(N-G)n-1,-1} = x^{-Gn-1} \left( c^n_{Nn} x^{Nn} + \dots + c^n_{Gn} x^{Gn} \right)$$

$$Q_{(N-G)n-1,-1} = c^{n}_{Nn}\, x^{(N-G)n-1} + \dots + c^{n}_{Gn}\, x^{-1} \qquad \text{II-123}$$

Using II-122 in II-104 gives us

$$I_1(n) = \int_0^{-A_1/A_2} Q_{(N-G)n-1,-1}\, \phi^{\{(N-G)n\}}\, dx$$

By parts

$$U = Q_{(N-G)n-1,-1} \qquad dV = \phi^{\{(N-G)n\}}\, dx$$

$$dU = Q'_{(N-G)n-1,-1}\, dx \qquad V = \phi^{\{(N-G)n-1\}}$$

Then

$$I_1(n) = \Big[ Q_{(N-G)n-1,-1}\, \phi^{\{(N-G)n-1\}} \Big]_0^{-A_1/A_2}$$

$$- \int_0^{-A_1/A_2} Q'_{(N-G)n-1,-1}\, \phi^{\{(N-G)n-1\}}\, dx$$

Using the expression for $\phi$ given by II-102, and performing the differentiation indicated in the bracket above, we have, using II-44' in II-111, a result similar to that for $I_2(n)$ in II-120, namely

$$I_1(n) = \Big[ Q_{(N-G)n-1,-1} \; g_{(N-G)n-1,0} \; x^{2Gn+u} \; (A_2 x + A_1) \Big]_0^{-A_1/A_2}$$

$$- \int_0^{-A_1/A_2} Q'_{(N-G)n-1,-1} \; \phi^{\{(N-G)n-1\}} \, dx \qquad \text{II-124}$$

where g is a polynomial in x of order and subdegree as indicated by the respective subscripts.

The bracket vanishes for all $n > 0$ ( $n = 0$ destroys the factor $(A_2 x + A_1)$ in $\phi$ as given by II-102) because of the factor $(A_2 x + A_1)$ and the presence of x to at least the power $2+u-1 = u+1 > 2$, since we have from II-47 that $u > 1$, and we know that $G \geqslant 1$.

If then we integrate by parts in II-104 a total of $(N-G)n$ times, we have, as an extension of II-124,

$$I_1(n) = (-1)^{(N-G)n} \int_0^{-A_1/A_2} Q^{\{(N-G)n\}}_{(N-G)n-1,-1} \; \phi \; dx$$

since each time we apply the method, the polynomial in the bracket such as we had in II-124 will be of the same order and subdegree as in II-124. By II-123 we then have

$$I_1(n) = (-1)^{(N-G)n} \, c_{Gn}^{n} \, (-1)^{(N-G)n} \, \left[(N-G)n\right]!$$

$$\cdot \int_0^{-A_1/A_2} x^{-1-(N-G)n} \, \phi \, dx$$

Using II-92 and II-102, this becomes, after replacing $(v-1)$ by $u$ from II-44', and after factoring out a $(-A_2)$ from $\phi$,

$$I_1(n) = (-1)^{2(N-G)n} \, A_1^{(N-G)n} \, \frac{\Gamma\left[(N+G)n+u\right]}{\Gamma(2Gn+u)} \, \left[(N-G)n\right]!$$

$$\cdot (-A_2)^{(N-G)n} \int_0^{-A_1/A_2} x^{(N+G)n+(u-1)-1-(N-G)n}$$

$$\cdot \left[\left(-\frac{A_1}{A_2}\right) - x\right]^{(N-G)n} dx$$

$$I_1(n) = (-A_1A_2)^{(N-G)n} \frac{\Gamma\left[(N+G)n+u\right]}{\Gamma(2Gn+u)} \left[(N-G)n\right]!$$

$$\cdot \int_0^{-A_1/A_2} x^{2Gn+u-2} \left[(-\frac{A_1}{A_2}) - x\right]^{(N-G)n} dx$$

Using II-76 and II-77, we have

$$I_1(n) = (-A_1A_2)^{(N-G)n} \frac{\Gamma\left[(N+G)n+u\right]}{\Gamma(2Gn+u)} \left[(N-G)n\right]!$$

$$\cdot \left(-\frac{A_1}{A_2}\right)^{(N+G)n+u-1} \frac{\Gamma(2Gn+u-1)\ \Gamma\left[(N-G)n+1\right]}{\Gamma\left[(N+G)n+u\right]}$$

Then

$$I_1(n) = (-1)^{u-1} \frac{A_1^{2Nn+u-1}}{A_2^{2Gn+u-1}} \frac{\left\{\left[(N-G)n\right]!\right\}^2}{2Gn+u-1} \qquad \text{II-125}$$

Using II-121 and II-125 in II-119, we have

$$I(n) = (-1)^{u} K_{2} \frac{A_{1}^{2Nn+u}}{A_{2}^{2Gn+u}} \frac{\left\{\left[(N-G)n\right]!\right\}^{2}}{2Nn+u}$$

$$+ (-1)^{u-1} K_{1} \frac{A_{1}^{2Nn+u-1}}{A_{2}^{2Gn+u-1}} \frac{\left\{\left[(N-G)n\right]!\right\}^{2}}{2Gn+u-1} \qquad \text{II-126}$$

$$= (-1)^{u} \frac{A_{1}^{2Nn+u-1}}{A_{2}^{2Gn+u-1}} \left\{\left[(N-G)n\right]!\right\}^{2} \left[\frac{A_{1}K_{2}}{A_{2}} \frac{1}{2Nn+u} - K_{1} \frac{1}{2Gn+u-1}\right] \qquad \text{II-127}$$

But from II-18 we have

$$N = \frac{u}{u-1} G$$

and

$$\frac{A_{1}K_{2}}{A_{2}} = \frac{u^{2}}{(u-1)^{2}} K_{1}$$

Using these in II-127, we have

$$I(n) = (-1)^{u} \frac{A_1^{2Nn+u-1}}{A_2^{2Gn+u-1}} \left\{\left[(N-G)n\right]!\right\}^2$$

$$\cdot \left[ \frac{u^2 K_1}{(u-1)^2} \cdot \frac{1}{2\left(\frac{u}{u-1}\right)Gn + u\left(\frac{u-1}{u-1}\right)} - \frac{K_1}{2Gn+u-1} \cdot \frac{u-1}{u-1} \right]$$

Hence we have as the integrated square formula

$$I(n) = \frac{(-1)^{u} K_1}{(u-1)(2Gn+u-1)} \frac{A_1^{2Nn+u-1}}{A_2^{2Gn+u-1}} \left\{\left[(N-G)n\right]!\right\}^2 \qquad \text{II-128}$$

From II-18 we have

$$G = \frac{u-1}{u} N$$

and

$$K_1 = \frac{(u-1)^2}{u^2} \frac{A_1}{A_2} K_2$$

Using these in II-128 gives us

$$I(n) = \frac{(-1)^{u} K_2}{(u-1)\left[2\left(\frac{u-1}{u}\right)Nn+(u-1)\left(\frac{u}{u}\right)\right]} \cdot \frac{(u-1)^2}{u^2} \cdot \frac{A_1^{2Nn+u}}{A_2^{2Gn+u}} \left\{\left[(N-G)n\right]!\right\}^2$$

Hence we have as an alternate form of the integrated square formula

$$I(n) = \frac{(-1)^{u} K_2}{u(2Nn+u)} \cdot \frac{A_1^{2Nn+u}}{A_2^{2Gn+u}} \left\{\left[(N-G)n\right]!\right\}^2 \qquad \text{II-129}$$

For n = o, II-129 becomes

$$I(o) = \frac{(-1)^{u} K_2}{u^2} \cdot \frac{A_1^{u}}{A_2^{u}} (0!)^2$$

$$= \frac{(-1)^{u-1}}{u(u-1)} \cdot \frac{A_1^{u-1}}{A_2^{u}} \left[-\frac{u-1}{u} A_1 K_2\right] \qquad \text{II-130}$$

which is seen to be the same as II-99, except for the bracket which is there

$$\left[A_2K_1(v-1) - A_1K_2(u-1)\right]$$

Replacing (v-1) by u from II-44, and multiplying by $\frac{u}{u}$, we have

$$\left[\frac{A_2K_1u^2 - A_1K_2u(u-1)}{u}\right]$$

But by the use of u for (v-1) in II-15, we have

$$A_2K_1u^2 = A_1K_2(u-1)^2$$

Using this in our previous result gives us the bracket of II-99 in the form

$$\left[\frac{A_1K_2(u-1)^2 - A_1K_2u(u-1)}{u}\right]$$

$$= \left[ \frac{\left[(u-1)-u\right](u-1)\ A_1 K_2}{u} \right]$$

$$= \left[ -\frac{u-1}{u} A_1 K_2 \right]$$

which is the same as the bracket in II-130.

Accordingly, we conclude that the formulas II-128 and II-129 for the integrated square are valid for all n, including $n = 0$.

## SECTION 11

## NON-TELESCOPING OF THE RECURSION PRODUCT OF THE POLYNOMIAL SOLUTIONS

Putting $(\phi+i)$ for $\phi$ in the denominator of the recursion product II-61 for the descending series solutions, we have

$$c^n_{k+\theta j} = \left(\frac{-1}{j}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} A_1 j^2 \phi^2 + 2A_1 j\phi k + (A_1 - A_2 K_1^*)k^2 + j(B_1 - A_1)\phi + \left[(B_1 - A_1) - K_1^*(B_2 - A_2)\right] k}{\prod_{\phi=1-i}^{\theta-i} (\phi+i) \left[A_2 j\phi + 2A_2 k + (A_2 j i + B_2 - A_2)\right]} c^n_k \qquad \text{II-131}$$

For $(\phi+i)$ to be a factor of the numerator for all n for some particular value of i, then $\phi = -i$ must make the numerator identically equal to zero for all values of k, since from I-45, $k = k(n)$, a function of n. That is, we must have

$$A_1 j^2(-i)^2 + 2A_1 j(-i)k + (A_1 - A_2 K_1^*)k^2$$
$$+ j(B_1-A_1)(-i) + \left[(B_1-A_1) - K_1^*(B_2-A_2)\right]k \overset{(k)}{=} 0$$

Therefore, the coefficients of the powers of k must all be zero. Then we have the equations

$k^2$ : $$A_1 - A_2 K_1^* = 0 \qquad \text{II-132}$$

$k$ : $$-2A_1 ji + B_1 - A_1 - K_1^*(B_2-A_2) = 0 \qquad \text{II-133}$$

$1$ : $$A_1 j^2 i^2 - B_1 ji + A_1 ji = 0 \qquad \text{II-134}$$

From II-132 we have

$$K_1^* = \frac{A_1}{A_2}$$

But from II-60 we had

$$K_1^* = \frac{K_1}{K_2} \qquad \text{II-60'}$$

Equating these two expressions for $K_1^*$ gives us

$$\frac{A_1}{A_2} = \frac{K_1}{K_2}$$

from which

$$\frac{A_1 K_2}{A_2 K_1} = 1 \qquad \text{II-135}$$

Using this in II-11 we have

$$k(n) = Gn$$

But we had

$$\dot{h}(n) = Gn \qquad \text{I-48'}$$

Hence

$$k(n) = h(n)$$

which is to say, the order and subdegree of the solutions are the same. Therefore, we have monomial solutions.

Using II-135 in II-18 gives us

$$\frac{v-1}{u-1} = 1$$

so that

$$v = u$$

But this violates the condition II-43 for the existence of a real interval of orthogonality.

Accordingly, we must conclude that for no value of i in II-131 will there be a telescoping of the recursion product such that the interval of orthogonality is preserved.

If for all n and some particular value of i, the numerator of II-131 is to have in common with the denominator the factor in the bracket of the denominator, then we must have

$$A_1 j^2 \phi^2 + 2A_1 j\phi k + (A_1 - A_2 K_1^*)k^2$$

$$+ j(B_1 - A_1)\phi + \left[(B_1 - A_1) - K_1^*(B_2 - A_2)\right]k$$

$$\overset{(\phi,k)}{\equiv} (E\phi + Fk + H)\left[A_2 j\phi + 2A_2 k + (A_2 ji + B_2 - A_2)\right] \qquad \text{II-136}$$

where E, F, H are constants to be determined. Hence the coefficients of the corresponding powers of $\phi$ and k must all be equal. This gives rise to the equations:

$\phi^2$ : $$A_1 j^2 = A_2 jE \qquad \text{II-137}$$

$\phi k$: $$2A_1 j = 2A_2 E + A_2 jF \qquad \text{II-138}$$

$k^2$ : $$A_1 - A_2 K_1^* = 2A_2 F \qquad \text{II-139}$$

$\phi$ : $$j(B_1 - A_1) = (A_2 ji + B_2 - A_2)E + A_2 jH \qquad \text{II-140}$$

k : $$(B_1 - A_1) - K_1^*(B_2 - A_2) = (A_2 ji + B_2 - A_2)F + 2A_2 H \qquad \text{II-141}$$

1 : $$0 = (A_2 ji + B_2 - A_2)H \qquad \text{II-142}$$

From II-137 we have that either $j = 0$, or

$$E = \frac{A_1 j}{A_2} \qquad \text{II-143}$$

We dismiss $j = o$ as giving monomial solutions.

Using II-143 in II-138 gives us

$$2A_1 j = 2A_2\left(\frac{A_1 j}{A_2}\right) + A_2 jF$$

from which

$$A_2 jF = 0 \qquad \text{II-144}$$

Hence, either $j = o$, $A_2 = 0$, or $F = 0$.

Again we dismiss $j = o$ as giving monomial solutions.

Using $A_2 = 0$ in II-139 we have that $A_1 = 0$. Therefore the differential equation has degenerated to first order, so that our orthogonality property is annihilated.

If from II-144 we take $F = 0$, then II-139 gives us

$$A_1 - A_2 K_1^* = 0 \qquad \text{II-132'}$$

from which we deduced II-135. Hence we conclude as we did following II-135, that for no value of i in II-136 will there be a telescoping of the recursion product such that the interval of orthogonality is preserved.

Using I-22 in I-33 we have as the recursion product for the ascending series solutions

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_2(k+\phi j)^2 + (B_2-A_2)(k+\phi j) + K_2\, L(n)}{\prod_{\phi=1}^{\theta} A_1(k+\phi j)^2 + (B_1-A_1)(k+\phi j) + K_1\, L(n)}\, c^n_k \qquad \text{II-145}$$

Solving I-61 for L(n), we have

$$L(n) = \frac{-A_1 k^2 - (B_1-A_1)k}{K_1} \qquad \text{II-146}$$

Using this and I-50 in II-145 and expanding, we have

$$c^n_{k+\theta j} = (-1)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_2 j^2 \phi^2 + 2A_2 j\phi k + A_2 k^2 + j(B_2-A_2)\phi + (B_2-A_2)k - K_2^* A_1 k^2 - K_2^*(B_1-A_1)k}{\prod_{\phi=1}^{\theta} A_1 j^2 \phi^2 + 2A_1 j\phi k + A_1 k^2 + j(B_1-A_1)\phi + (B_1-A_1)k - A_1 k^2 - (B_1-A_1)k} c^n_k$$

$$= \left(\frac{-1}{j}\right)^\theta \frac{\prod_{\phi=0}^{\theta-1} A_2 j^2 \phi^2 + 2A_2 j\phi k + (A_2 - A_1 K_2^*)k^2 + j(B_2-A_2)\phi + \left[(B_2-A_2) - K_2^*(B_1-A_1)\right]k}{\prod_{\phi=1}^{\theta} \phi \left[A_1 j\phi + 2A_1 k + (B_1-A_1)\right]} c^n_k \qquad \text{II-147}$$

The problem of the telescoping of this recursion product for the ascending series solutions is then the same as that of the telescoping of the recursion product for the descending series solutions II-131, as is seen by merely interchanging the subscripts of the coefficients of the above recursion product. Hence, the methods applied to the study of II-131 and the results obtained there, hold here also.

## SECTION 12

## SUMMARY OF RESULTS

$$(A_1x^2 + A_2x^{-j+2})y''_n + (B_1x + B_2x^{-j+1})y'_n + L(n)\left[K_1 + K_2x^{-j}\right]y_n = 0 \qquad \begin{matrix}\text{I-1}\\ \text{I-22}\\ \text{I-27}\end{matrix}$$

$$u = \frac{B_1}{A_1}, \quad v = \frac{B_2}{A_2}, \quad v > u > 1, \quad (v-u-1) = 0 \text{ for } I(n) \text{ to hold.} \qquad \begin{matrix}\text{II-9}\\ \text{II-10}\\ \text{II-44}\end{matrix}$$

$$A_2K_1(v-1)^2 = A_1K_2(u-1)^2; \quad \frac{A_1}{A_2} < 0 \text{ if } u \text{ fractional or } j \text{ even.} \qquad \begin{matrix}\text{II-15}\\ \text{II-41}\\ \text{II-98+}\end{matrix}$$

$$\overset{+}{A}_1,\ \overset{-}{A}_2,\ \overset{+}{B}_1,\ \overset{-}{B}_2,\ \overset{\mp}{K}_1,\ \overset{\pm}{K}_2 \text{ for interval } \left(0, -\frac{A_1}{A_2}\right), \quad L(n) \gtrless 0 \qquad \begin{matrix}\text{II-55}\\ \text{II-56}\end{matrix}$$

$$\frac{N}{G} = \frac{v-1}{u-1}, \quad k(n) = Nn, \quad h(n) = Gn, \quad G = \left|\frac{K_1A_2(B_1-A_1)j}{Q}\right| \qquad \begin{matrix}\text{II-18}\\ \text{II-13}\\ \text{I-48}\\ \text{II-34}\end{matrix}$$

$$Q = \text{g.c.d. } JK_1, \ A_2(B_1-A_1)j\sqrt{|K_1E|}, \ A_2(B_1-A_1)Fj \qquad \text{II-33}$$

$$J = \text{g.c.d. } A_2(B_1-A_1)j, \ \begin{vmatrix} B_1 & B_2 \\ A_1 & A_2 \end{vmatrix} \qquad \text{II-26}$$

$$E = \text{g.c.d. } A_1, \ K_1; \quad F = \text{g.c.d. } (B_1-A_1), \ K_1 \qquad \begin{matrix}\text{II-27}\\ \text{II-30}\end{matrix}$$

$$L(n) = \frac{-A_2 k^2(n) - (B_2 - A_2)\, k(n)}{K_2} = \frac{-A_1 h^2(n) - (B_1 - A_1)\, h(n)}{K_1} \qquad \begin{matrix}\text{II-48}\\ \text{II-49}\end{matrix}$$

$$W = x^{u-2} \left(A_1 + A_2 x^{-j}\right)^{\frac{u-v}{j} - 1} \left(K_1 + K_2 x^{-j}\right) \qquad \text{II-45}$$

$$y_n = c_k^n \sum_{\theta=0}^{(\frac{G-N}{j})n} \left(-\frac{A_1}{A_2 j}\right)^{\theta} \frac{\prod_{\phi=0}^{\theta-1} \left[(N-G)n + j\phi\right]\left[(N+G)n + (u-1) + j\phi\right]}{\theta!\, \prod_{\phi=1}^{\theta} \left[2Nn + (v-1) + j\phi\right]} \, x^{Nn+\theta j} \qquad \begin{matrix}\text{II-71}\\ \text{II-68}\\ \text{II-67}\end{matrix}$$

$$c_k^n = A_2^{(N-G)n} \frac{\Gamma\left[2Nn + (v-1)\right]}{\Gamma\left[(N+G)n + u\right]}$$

When Derivative Definition is used, then $j = -1$. II-90

$$y_n = x^{-Gn-u+1} D^{(N-G)n+(v-u-1)} \left[x^{(N+G)n+(v-2)} A(A_2 x + A_1)^{(N-G)n}\right] \qquad \text{II-95}$$

$$I(n) = \frac{(-1)^u K_2}{u(2Nn + u)} \frac{A_1^{2Nn+u}}{A_2^{2Gn+u}} \left\{\left[(N-G)n\right]!\right\}^2 \qquad \text{II-129}$$

## SECTION 13

## NUMERICAL EXAMPLES

EXAMPLE 1. $u, v$ integers; $G=1$; constant of proportionality = 1.

$$A_1 = 1, \quad A_2 = -1, \quad B_1 = 2, \quad B_2 = -3, \quad K_1 = -1, \quad K_2 = 4, \quad j = -1.$$

$u = \frac{2}{1} = 2$, $v = \frac{-3}{-1} = 3$. With these II-10,-15,-44,-55 all hold and $\frac{A_1}{A_2} = \frac{1}{-1} = -1$ gives interval $(0,1)$. $\frac{N}{G} = \frac{2}{1}$, so that taking $G=1$, $N=2$, we have $k(n) = 2n$, $h(n) = n$, and

$$L(n) = \frac{-(1)n^2 - (2-1)n}{-1} = n^2 + n = n(n+1)$$

Therefore the differential equation

$$(x^2 - x^3)y''_n + (2x - 3x^2)y'_n + n(n+1)(-1 + 4x)\, y_n = 0$$

which is self-adjoint, has polynomial solutions

$$y_n = (-1)^n \frac{(4n+1)!}{(3n+1)!} \sum_{\theta=0}^{n} \frac{(-1)^\theta}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (n-\phi)(3n+1-\phi)}{\prod_{\phi=1}^{\theta} (4n+2-\phi)} x^{2n-\theta}$$

orthogonal in $(0,1)$, with weight function

$$W = x^{0} (1-x)^{0} (-1+4x) = 4x-1$$

having the derivative definition

$$y_n = x^{-n-1} D^{n}\left[x^{3n+1} (1-x)^{n}\right]$$

and integrated square formula

$$I(n) = \frac{(-1)^{2} \; 4}{2(4n+2)} \; \frac{1^{4n+2}}{(-1)^{2n+2}} \; (n!)^{2} = \frac{(n!)^{2}}{2n+1}$$

$y_c = 1 \qquad I(o) = 1 \qquad I(1) = \frac{1}{3} \qquad I(2) = \frac{(2!)^{2}}{2(2)+1} = \frac{4}{5}$

$$y_1 = x^{-2} D^{1}\left[x^{4} (1-x)\right] = x^{-2} D(x^{4} - x^{5}) = x^{-2} (4x^{3} - 5x^{4}) = 4x - 5x^{2}$$

$$y_2 = x^{-3} D^{2}\left[x^{7} (1-x)^{2}\right] = x^{-3} D^{2} (x^{7} - 2x^{8} + x^{9})$$

$$= x^{-3} (9 \cdot 8x^{7} - 2 \cdot 8 \cdot 7x^{6} + 7 \cdot 6x^{5}) = 72x^{4} - 112x^{3} + 42x^{2}$$

EXAMPLE 2. u,v fractions; G=1; constant of proportionality = 1

$A_1 = 2,\quad A_2 = -2,\quad B_1 = 3,\quad B_2 = -5,\quad K_1 = -1,\quad K_2 = 9,\quad j = -1$

$u = \frac{3}{2},\quad v = \frac{-5}{-2} = \frac{5}{2}$. With these II-10,-15,-44,-55 all hold

and $\frac{A_1}{A_2} = \frac{2}{-2} = -1 < 0$, as required for u fractional, gives

interval (0,1). $\frac{N}{G} = \frac{3}{2} \Big/ \frac{1}{2} = \frac{3}{1}$, so that taking $G=1$, $N=3$,

we have $k(n) = 3n$, $h(n) = n$, and

$$L(n) = \frac{-(2)n^2 - (3-2)n}{-1} = 2n^2 + n = n(2n+1)$$

Therefore the differential equation

$$(2x^2 - 2x^3)y_n'' + (3x - 5x^2)y_n' + n(2n+1)(-1 + 9x)y_n = 0$$

has polynomial solutions

$$y_n = (-2)^{2n} \frac{\Gamma\left[6n + \frac{3}{2}\right]}{\Gamma\left[4n + \frac{3}{2}\right]} \sum_{\theta=0}^{2n} \frac{(-1)^\theta}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (2n-\phi)(4n + \frac{1}{2} - \phi)}{\prod_{\phi=1}^{\theta} (6n + \frac{3}{2} - \phi)} x^{3n-\theta}$$

$$y_n = 2^{2n} \frac{\Gamma\left[6n+\frac{3}{2}\right]}{\Gamma\left[4n+\frac{3}{2}\right]} \sum_{\theta=0}^{2n} \frac{(-1)^{\theta}}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (2n-\phi)(8n+1-2\phi)}{\prod_{\phi=1}^{\theta} (12n+3-2\phi)} x^{3n-\theta}$$

orthogonal in (0,1), with weight function

$$W = x^{-1/2} (2-2x)^{0} (-1+9x) = 9x^{1/2} - x^{-1/2}$$

having the derivative definition

$$y_n = x^{-n-1/2} D^{2n} \left[ x^{4n+1/2} (2-2x)^{2n} \right]$$

$$= 2^{2n} x^{-n-1/2} D^{2n} \left[ x^{4n+1/2} (1-x)^{2n} \right]$$

and integrated square formula

$$I(n) = \frac{(-1)^{3/2}\ 9}{3/2\ (6n+3/2)} \frac{2^{6n+3/2}}{(-2)^{2n+3/2}} \left[(2n)!\right]^2 = \frac{2^{4n+2}}{4n+1} \left[(2n)!\right]^2$$

EXAMPLE 3. u, v integers; $G \neq 1$; constant of proportionality = 1

$A_1 = 2$, $A_2 = -1$, $B_1 = 6$, $B_2 = -4$, $K_1 = -8$, $K_1 = 9$, $j = -1$

$u = \frac{6}{2} = 3$, $v = \frac{-4}{-1} = 4$. With these II-10, -15, -44, -55 all hold, and $\frac{A_1}{A_2} = \frac{2}{-1} = -2$ gives interval (0, 2). $\frac{N}{G} = \frac{3}{2}$, so that taking $G = 2$, $N = 3$, we have $k(n) = 3n$, $h(n) = 2n$, and

$$L(n) = \frac{-(2)(2n)^2 - (6-2)(2n)}{-8} = n^2 + n = n(n+1)$$

Therefore the differential equation

$$(2x^2 - x^3)y_n'' + (6x - 4x^2)y_n' + n(n+1)(-8 + 9x)y_n = 0$$

has polynomial solutions

$$y_n = (-1)^n \frac{(6n+2)!}{(5n+2)!} \sum_{\theta=0}^{n} \frac{(-2)^\theta}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (n-\phi)(5n+2-\phi)}{\prod_{\phi=1}^{\theta} (6n+3-\phi)} x^{3n-\theta}$$

orthogonal in $(0,2)$, with weight function

$$W = x(2-x)^{0}\,(-8+9x) = 9x^{2} - 8x = x(9x-8)$$

having the derivative definition

$$y_n = x^{-2n-2}\,D^{n}\left[x^{5n+2}\,(2-x)^{n}\right]$$

and integrated square formula

$$I(n) = \frac{(-1)^{3}\;9}{3\,(6n+3)}\;\frac{2^{6n+3}}{(-1)^{4n+3}}\;(n!)^{2} = \frac{2^{6n+3}\,(n!)^{2}}{2n+1}$$

EXAMPLE 4. u,v fractions; $G \neq 1$; constant of proportionality = 1

$A_1 = 2$, $A_2 = -2$, $B_1 = 5$, $B_2 = -7$, $K_1 = -9$, $K_2 = 25$, $j = -1$

$u = \frac{5}{2}$, $v = \frac{-7}{-2} = \frac{7}{2}$ With these II-10,-15,-44,-55 all hold,

and $\frac{A_1}{A_2} = \frac{2}{-2} = -1 < 0$, as required for u fractional, gives

interval (0,1). $\frac{N}{G} = \frac{5}{2} / \frac{3}{2} = \frac{5}{3}$, so that taking $G=3$, $N=5$,

we have $k(n) = 5n$, $h(n) = 3n$, and

$$L(n) = \frac{-(2)(3n)^2 - (5-2)(3n)}{-9} = 2n^2 + n = n(2n+1)$$

Therefore the differential equation

$$(2x^2 - 2x^3)y_n'' + (5x - 7x^2)y_n' + n(2n+1)(-9 + 25x)y_n = 0$$

has polynomial solutions

$$y_n = (-2)^{2n} \frac{\Gamma\left[10n + \frac{5}{2}\right]}{\Gamma\left[8n + \frac{5}{2}\right]} \sum_{\theta=0}^{2n} \frac{(-1)^\theta}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (2n-\phi)(8n + \frac{3}{2} - \phi)}{\prod_{\phi=1}^{\theta} (10n + \frac{5}{2} - \phi)} x^{5n-\theta}$$

$$y_n = 2^{2n} \frac{\Gamma\left[10n + \frac{5}{2}\right]}{\Gamma\left[8n + \frac{5}{2}\right]} \sum_{\theta=0}^{2n} \frac{(-1)^{\theta}}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (2n-\phi)(16n+3-2\phi)}{\prod_{\phi=1}^{\theta} (20n+5-2\phi)} x^{5n-\theta}$$

orthogonal in (0,1), with weight function

$$W = x^{1/2} (2-2x)^{0} (-9+25x) = 25x^{3/2} - 9x^{1/2}$$

having the derivative definition

$$y_n = x^{-3n-3/2} D^{2n}\left[x^{8n+3/2} (2-2x)^{2n}\right]$$

$$= 2^{2n} x^{-3n-3/2} D^{2n}\left[x^{8n+3/2} (1-x)^{2n}\right]$$

and integrated square formula

$$I(n) = \frac{(-1)^{5/2} \quad 25}{5/2 \; (10n+5/2)} \; \frac{2^{10n+5/2}}{(-2)^{6n+5/2}} \left[(2n)!\right]^2 = \frac{2^{4n+2}}{4n+1} \left[(2n)!\right]^2$$

EXAMPLE 5. u, v integers; constant of proportionality $\neq 1$

$A_1 = 3$, $A_2 = -1$, $B_1 = 9$, $B_2 = -4$, $K_1 = -8$, $K_2 = 6$, $j = -1$

$u = \frac{9}{3} = 3$, $v = \frac{-4}{-1} = 4$. With these II-10, -15, -44, -55 all hold, and $\frac{A_1}{A_2} = \frac{3}{-1} = -3$ gives interval (0, 3). $\frac{N}{G} = \frac{3}{2}$ so that if we try $G=2$, $N=3$, we have $k(n) = 3n$, $h(n) = 2n$, and

$$L(n) = \frac{-(3)(2n)^2 - (9-3)(2n)}{-8} = \frac{3n^2 + 3n}{2}$$

which is not an integer. But by inspection of this result for L(n), we see that if we augment the above trial values of N and G by a constant of proportionality = 2, we will have $G=4$, $N=6$, $k(n) = 6n$, $h(n) = 4n$, and

$$L(n) = \frac{-(3)(4n)^2 - (9-3)(4n)}{-8} = 6n^2 + 3n = 3n(2n+1)$$

an integer. Therefore the differential equation

$$(3x^2 - x^2)y''_n + (9x - 4x^2)y'_n + 3n(2n+1)(-8 + 6x)y_n = 0$$

has polynomial solutions

$$y_n = \frac{(12n+2)!}{(10n+2)!} \sum_{\theta=0}^{2n} \frac{(-3)^\theta}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (2n-\phi)(10n+2-\phi)}{\prod_{\phi=1}^{\theta} (12n+3-\phi)} x^{6n-\theta}$$

orthogonal in $(0,3)$, with weight function

$$W = x(3-x)^0 (-8+6x) = 6x^2 - 8x = 2x(3x-4)$$

having the derivative definition

$$y_n = x^{-4n-2} D^{2n}\left[x^{10n+2} (3-x)^{2n}\right]$$

and integrated square formula

$$I(n) = \frac{(-1)^3 \; 6}{3(12n+3)} \; \frac{3^{12n+3}}{(-1)^{8n+3}} \left[(2n)!\right]^2 = \frac{2\;(3)^{12n+2}}{4n+1} \left[(2n)!\right]^2$$

EXAMPLE 6. u, v fractions; constant of proportionality $\neq 1$

$A_1 = 3$, $A_2 = -3$, $B_1 = 5$, $B_2 = -8$, $K_1 = -12$, $K_2 = 75$, $j = -1$

$u = \frac{5}{3}$, $v = \frac{-8}{-3} = \frac{8}{3}$. With these II-10,-15,-44,-55 all hold, and $\frac{A_1}{A_2} = \frac{3}{-3} = -1 < 0$, as required for u fractional, gives interval (0,1). $\frac{N}{G} = \frac{5}{3} / \frac{2}{3} = \frac{5}{2}$, so that if we try $G=2$, $N=5$, we have $k(n) = 5n$, $h(n) = 2n$, and

$$L(n) = \frac{-(3)(2n)^2 - (5-3)(2n)}{-12} = \frac{3n^2 + n}{3}$$

which is not an integer. But by inspection of this result for $L(n)$, we see that if we augment the above trial values of N and G by a constant of proportionality 3, then we will have $G=6$, $N=15$, $k(n) = 15n$, $h(n) = 6n$, and

$$L(n) = \frac{-(3)(6n)^2 - (5-3)(6n)}{-12} = 9n^2 + n = n(9n+1)$$

an integer. Therefore the differential equation

$$(3x^2 - 3x^3)y_n'' + (5x - 8x^2)y_n' + n(9n+1)(-12 + 75x)y_n = 0$$

has polynomial solutions

$$y_n = (-3)^{9n} \frac{\Gamma\left[30n + \frac{5}{3}\right]}{\Gamma\left[21n + \frac{5}{3}\right]} \sum_{\theta=0}^{9n} \frac{(-1)^{\theta}}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (9n-\phi)(21n + \frac{2}{3} - \phi)}{\prod_{\phi=1}^{\theta} (30n + \frac{5}{3} - \phi)} x^{15n-\theta}$$

$$= (-3)^{9n} \frac{\Gamma\left[30n + \frac{5}{3}\right]}{\Gamma\left[21n + \frac{5}{3}\right]} \sum_{\theta=0}^{9n} \frac{(-1)^{\theta}}{\theta!} \frac{\prod_{\phi=0}^{\theta-1} (9n-\phi)(63n + 2 - 3\phi)}{\prod_{\phi=1}^{\theta} (90n + 5 - 3\phi)} x^{15n-\theta}$$

orthogonal in (0,1), with weight function

$$W = x^{-1/3}(3 - 3x)^{0}(-12 + 75x) = 3x^{-1/3}(25x - 4)$$

having the derivative definition

$$y_n = x^{-6n - 2/3} D^{9n}\left[x^{21n + 2/3} (3 - 3x)^{9n}\right]$$

$$= 3^{9n} x^{-6n - 2/3} D^{9n}\left[x^{21n + 2/3} (1 - x)^{9n}\right]$$

and integrated square formula

$$I(n) = \frac{(-1)^{5/3} 75}{5/3\ (30n + 5/3)} \frac{3^{30n + 5/3}}{(-3)^{12n + 5/3}} \left[(9n)!\right]^2 = \frac{3^{18n+3}}{18n + 1} \left[(9n)!\right]^2$$

# PART III

# NON-CLASSICAL ORTHOGONAL POLYNOMIAL SOLUTIONS
with
TRANSCENDENTAL WEIGHT FUNCTION

## SECTION 1

## ORDER AND SUBDEGREE OF THE POLYNOMIAL SOLUTIONS

In this third part of the dissertation we consider the polynomial solutions which obtain when I-52 is satisfied by taking

$$A_2 = 0 \qquad \text{III-1}$$

Using this in the indicial equation for the descending series solutions, I-45, and solving for $k(n)$, we have

$$k(n) = -\frac{K_2\, L(n)}{B_2} \qquad \text{III-2}$$

Using in this the expression for $L(n)$ given by I-49, we have for the order of the descending series solutions

$$k(n) = \frac{A_1K_2G^2}{B_2K_1}n^2 + \frac{K_2(B_1-A_1)G}{B_2K_1}n \qquad \text{III-3}$$

Let

$$M = \frac{A_1K_2G^2}{B_2K_1}, \qquad N = \frac{K_2(B_1-A_1)G}{B_2K_1} \qquad \text{III-4}$$

Then III-3 may be written

$$k(n) = Mn^2 + Nn \qquad \text{III-5}$$

It is clear that $k(n)$ will be an integer for all $n$ only if $M$ and $N$ are integers.

At this point it is convenient to interject the substitutions, analagous to those of II-9,

$$u = \frac{B_1}{A_1}, \qquad v = \frac{B_2}{A_1} \qquad \text{III-6}$$

and to state that it will be shown later following III-38 that the existence and non-vanishing of the integrated square of $y_0$ requires that

$$u > 1 \qquad \text{III-7}$$

## SECTION 2

## COMPATIBILITY OF ORDER, SUBDEGREE, JUMP, AND PARAMETER

For compatibility of order $k(n)$, subdegree $h(n)$, and jump $j$ of the descending polynomial solutions, we must have for all $n$

$$\frac{k(n) - h(n)}{j} = \text{an integer} < 0$$ III-8 (II-19')

Using I-48 and III-5, this requirement becomes

$$\frac{M}{j} n^2 + \frac{N-G}{j} n = \text{an integer} < 0$$ III-9

This means that we must have

$$\frac{M}{j} = -a$$ III-10

where a is a positive integer. With the aid of III-4 we write this requirement as

$$\frac{G^2}{j} \left[ \frac{A_1 K_2}{B_2 K_1} \right] = -a$$ III-11

Equation III-9 also dictates that we have

$$\frac{N-G}{j} = -b$$ III-12 (II-21')

where b is a positive integer. It will be shown later in III-80 that for the non-varishing of the integrated square of $y_0$ we must have

$$b \neq 0 \qquad \text{III-13}$$

Equation III-12 then implies

$$N \neq G \qquad \text{III-14}$$

Using III-4, equation III-12 becomes

$$\frac{1}{j}\left[\frac{K_2(B_1-A_1)G}{B_2K_1} - G\right] = -b$$

which we write as

$$\frac{G}{j}\left[\frac{K_2(B_1-A_1)}{B_2K_1} - 1\right] = -b \qquad \text{III-15}$$

Combining the terms in the bracket, we have

$$\frac{G}{j}\left[\frac{K_2(B_1-A_1) - B_2K_1}{B_2K_1}\right] = -b \qquad \text{III-16}$$

We may write this in determinant form as

$$\frac{G\begin{vmatrix} K_1 & K_2 & 0 \\ B_1 & B_2 & 1 \\ A_1 & 0 & 1 \end{vmatrix}}{B_2K_1j} = b \qquad \text{III-17}$$

Finally, we must of course have

$$k(n) \geqslant h(n) \geqslant o \qquad \text{III-18}$$

Using I-48 and III-5 this becomes

$$Mn^2 + Nn \geqslant Gn \qquad \text{III-19}$$

Taking $n = 1$, we have the requirement as

$$M + N \geqslant G \qquad \text{III-20}$$

Now from III-11 we see that $G^2$ must contain $\frac{B_2K_1j}{H}$ where

$$H = \text{g.c.d. of } A_1K_2 \text{ and } B_2K_1j \qquad \text{III-21}$$

That is, G must contain

$$\sqrt{\left|\frac{B_2K_1j}{H}\right|} \qquad \text{III-22}$$

From III-17 we see also that G must contain

$$\frac{B_2K_1j}{J} \qquad \text{III-23}$$

where

$$J = \text{g.c.d. of } B_2K_1j \text{ and } \begin{vmatrix} K_1 & K_2 & 0 \\ B_1 & B_2 & 1 \\ A_1 & 0 & 1 \end{vmatrix} \qquad \text{III-24}$$

or, by III-15, if $|j| = 1$, then

$$J = \text{g.c.d. of } B_2K_1 \text{ and } K_2(B_1-A_1)$$

The closure equation solved for the parameter L(n) is

$$L(n) = \frac{-A_1G^2n^2 - (B_1-A_1)\ Gn}{K_1} \qquad \text{I-49''}$$

From this we see that for L(n) to be an integer for all n, then the coefficients of both powers of n must be integers. Hence $G^2$ must contain $\frac{K_1}{E}$, where

$$E = \text{g.c.d. of } A_1 \text{ and } K_1 \qquad \text{II-27'}$$

That is, G must contain

$$\sqrt{\left|\frac{K_1}{E}\right|} \qquad \text{II-28'}$$

Also, G must contain

$$\frac{K_1}{F} \qquad \text{II-29'}$$

where

$$F = \text{g.c.d. of } (B_1 - A_1) \text{ and } K_1 \qquad \text{II-30}'$$

Accordingly,

$$G^* = \left| B_2 K_1 j \right| \qquad \text{III-25}$$

will certainly contain III-22, III-23, II-28', and II-29', and with a surplus as indicated by the brackets in the following equations:

$$G^* = \left| \sqrt{\left| \frac{B_2 K_1 j}{H} \right|} \left[ \sqrt{\left| B_2 K_1 H j \right|} \right] \right|$$

$$= \left| \frac{B_2 K_1 j}{J} \quad [J] \right|$$

$$= \left| \sqrt{\left| \frac{K_1}{E} \right|} \left[ B_2 j \sqrt{\left| K_1 E \right|} \right] \right|$$

$$= \left| \frac{K_1}{F} \left[ B_2 F j \right] \right|$$

Hence, the least value of G which will make III-11, III-17, and I-49" integers is

$$G = \frac{G^*}{Q} \qquad \text{III-26}$$

where

$$Q = \text{g.c.d. of } \sqrt{|B_2K_1HJ|},\ J,\ B_2j\sqrt{|K_1E|},\ \text{and } B_2Fj \qquad \text{III-27}$$

That is,

$$G = \left|\frac{B_2K_1j}{Q}\right| \qquad \text{III-28}$$

where Q, H, J, E, F, are given by III-27, III-21, III-24, II-27', and II-30', respectively.

Such choice of G as indicated in III-28 will also satisfy the condition III-20. For using this value of G in III-4 and the result in III-20 gives us

$$\frac{A_1K_2}{B_2K_1}\left|\frac{B_2K_1j}{Q}\right|^2 + \frac{K_2(B_1-A_1)}{B_2K_1}\left|\frac{B_2K_1j}{Q}\right| \geqslant \left|\frac{B_2K_1j}{Q}\right|$$

as the condition which must hold.

For $\frac{B_2K_1j}{Q} > 0$, with $j < o$, and taking $H > 0$, $Q > 0$, $J > 0$, we have

$$\frac{A_1K_2}{B_2K_1}\left(\frac{B_2K_1j}{Q}\right)^2 + \frac{K_2(B_1-A_1)}{B_2K_1}\left(\frac{B_2K_1j}{Q}\right) \geq \frac{B_2K_1j}{Q}$$

We may write this as

$$j\left(\frac{A_1K_2}{H}\right)\frac{\sqrt{B_2K_1Hj}}{Q}\;\frac{\sqrt{B_2K_1Hj}}{Q} + j\left[\frac{K_2(B_1-A_1) - B_2K_1}{J}\right]\frac{J}{Q} \geq 0 \qquad \text{III-29}$$

By III-27, $\frac{J}{Q}$ is an integer. By III-24, the bracket and $\frac{B_2K_1j}{J}$ are integers. Hence, $\frac{K_2(B_1-A_1)j}{J}$ is an integer.

Furthermore, by III-21 and III-27 the ratios in the first term of III-29 are integers. Hence the condition III-29 becomes

$$\frac{A_1K_2j}{Q}\left(\frac{B_2K_1j}{Q}\right) + j\,\frac{K_2(B_1-A_1)}{J}\;\frac{J}{Q} \geq \frac{B_2K_1j}{Q} \qquad \text{III-30}$$

where the sign compatibility conditions III-46, III-47, to be discussed later, together with III-7 in III-6, indicate that the above ratios are all positive (integers as just shown).

The left member thus indicates that the integer $\frac{B_2K_1j}{Q}$, which is G, is multiplied by a positive integer, and to this is added another positive integer. The result is then certainly greater than G itself, which appears in the right member. Hence the inequality holds, and III-20 is satisfied.

For $\frac{B_2K_1j}{Q} < 0$, taking j, H, Q, J as before, we have instead of III-30 the result

$$\frac{A_1K_2j}{Q}\left(\frac{B_2K_1j}{Q}\right) - j\,\frac{K_2(B_1-A_1)}{J}\;\frac{J}{Q} \geqslant -\,\frac{B_2K_1j}{Q}$$

where the sign compatibility conditions III-45, III-48, to be discussed later, insure that the indicated ratios are all positive (integers, as shown above), so that we may conclude as before that the inequality holds and that III-20 is satisfied.

Thus we have shown that it is always possible to find polynomial solutions if III-1 and III-7 hold, which, if III-14 also holds, have non-vanishing integrated square for all n.

To obtain the values of M, N, G, it may, however, be more expedient to use III-4 to get the ratio $\frac{N}{G}$ , and then to proceed by inspection to find the precise value of G required to make M and L(n) integers, as will be illustrated by the numerical examples of section 13, rather than to follow the formal procedure outlined above.

## SECTION 3

## THE INTEGRATING FACTOR

Using $A_2 = 0$ from III-1 in II-35, we have

$$wp = e^{\int \frac{B_1 x^{r+1} + B_2 x^{s+1}}{A_1 x^{r+2}} dx} \qquad \text{III-31}$$

Dividing out an $x^{r+1}$ and using III-6, we have

$$wp = e^{\int (\frac{u}{x} + v x^{s-r-1})\, dx}$$

Using $j = r-s < 0$ from I-27, as for the descending series solutions, and integrating, we have

$$wp = e^{u \ln x + v \frac{x^{-j}}{-j}}$$

$$= x^{u} e^{-\frac{v}{j} x^{-j}} \qquad \text{III-32}$$

Using $A_2 = 0$ from III-1 in I-2, and the result in III-32 above, we have as the integrating factor

$$W = \frac{x^{u} e^{-\frac{v}{j} x^{-j}}}{A_1 x^{r+2}}$$

$$= \frac{1}{A_1} x^{u-r-2} e^{-\frac{v}{j} x^{-j}} \qquad \text{III-33}$$

## SECTION 4

## THE INTERVAL OF ORTHOGONALITY

From III-32 wp will have real roots at

$x = 0, (\pm 1)^{-j+1} \infty$ if $v < 0 < u$ III-34

$x = 0, -\infty$ if $u, v > 0$ $j$ odd III-35

$x = \pm \infty$ if $u = 0, \ v < 0$ $j$ even III-36

The possibility III-36 of only infinite roots is, however, untenable, since we cannot have $u = 0$, but must have, as stated previously,

$$u > 1 \qquad \text{III-7}'$$

## SECTION 5

## THE WEIGHT FUNCTION

Using III-33 in I-24 gives us

$$W = \frac{1}{A_1} x^{u-r-2} e^{-\frac{v}{j} x^{-j}} (K_1 x^r + K_2 x^s)$$

Combining with the first factor an $x^r$ taken from the last factor, and using I-27, we have as the weight function for the descending series solutions

$$W = \frac{1}{A_1} x^{u-2} e^{-\frac{v}{j} x^{-j}} (K_1 + K_2 x^{-j}) \qquad \text{III-37}$$

Now from III-5 and I-48 we have

$$k(o) = h(o) = o$$

so that $y_o$ is a constant. Accordingly, using III-37 in I-25, with $m = n = o$, we have for the integrated square of $y_o$

$$\frac{y_o^2}{A_1} \int_{x_o}^{x_1} x^{u-2} e^{-\frac{v}{j} x^{-j}} (K_1 + K_2 x^{-j})\, dx \qquad \text{III-38}$$

where, no matter which of the intervals III-34, III-35, or III-36 is considered, the origin is included in the range of integration.

The product of the binomial and the expansion of the exponential begins with a constant. The product of this constant and the factor $x^{u-2}$ must be $x$ to a power greater than $(-1)$, if the integral is to converge at the origin. That is, we must have

$$u-2 > -1$$

which means that

$$u > 1 \qquad \text{III-7"}$$

## SECTION 6

## COMPATIBILITY OF SIGNS OF THE COEFFICIENTS IN THE DIFFERENTIAL EQUATION

From III-2, the indicial equation for the descending series solutions, solved for $L(n)$, is

$$L(n) = -\frac{B_2}{K_2}k(n) \qquad \text{III-39}$$

The closure equation solved for $L(n)$ is

$$L(n) = \frac{-A_1h^2(n) - (B_1-A_1)\,h(n)}{K_1} \qquad \text{II-49}'$$

Since for polynomials we must have $k(n) \geq h(n) \geq 0$ for all n, then from these last two equations and the fact that $A_1 > 0$ by I-1 ff., we conclude that

$$\begin{aligned} &\text{For } L(n) \geq 0, \text{ then } \overset{+}{A_1}, \overset{-}{K_1} \text{ have opposite signs} \\ &\qquad\qquad\qquad\ \ \text{and } B_2, K_2 \text{ have opposite signs} \qquad \text{III-40} \end{aligned}$$

For $L(n) \leq 0$, then $\overset{+}{A}_1$, $\overset{+}{K}_1$ have the same signs
and $B_2$, $K_2$ have the same signs. III-41

From III-6 and III-7, for permissible intervals III-34 and III-35, we conclude that

$u > 0$, and $\overset{+}{A}_1$, $\overset{+}{B}_1$ have the same signs. III-42

For the interval $[0, (\pm 1)^{-j+1} \infty]$, then
$v < 0$, and $\overset{+}{A}_1$, $\overset{-}{B}_2$ have opposite signs. III-43

For the interval $(0, -\infty)$, with $j$ odd, then
$v > 0$, and $\overset{+}{A}_1$, $\overset{+}{B}_2$ have the same signs. III-44

Accordingly, from III-40, III-42, III-43 we see that for the interval $[0, (\pm 1)^{-j+1} \infty]$, with $L(n) \geq 0$, then we must have

$$\overset{+}{A}_1, \overset{+}{B}_1, \overset{-}{B}_2, \overset{-}{K}_1, \overset{+}{K}_2 \qquad \text{III-45}$$

From III-41, III-42, III-43 we see that for the interval $\left[0,\ (\pm 1)^{-j+1}\infty\right]$, with $L(n) \leqslant 0$, then we must have

$$\overset{+}{A}_1,\ \overset{+}{B}_1,\ \overset{-}{B}_2,\ \overset{+}{K}_1,\ \overset{-}{K}_2 \qquad \text{III-46}$$

From III-40, III-42, III-44 we see that for the interval $(0, -\infty)$, with $j$ odd, and $L(n) \geqslant 0$, then we must have

$$\overset{+}{A}_1,\ \overset{+}{B}_1,\ \overset{+}{B}_2,\ \overset{-}{K}_1,\ \overset{-}{K}_2 \qquad \text{III-47}$$

From III-41, III-42, III-44 we see that for the interval $(0, -\infty)$, with $j$ odd, and $L(n) \leqslant 0$, then we must have

$$\overset{+}{A}_1,\ \overset{+}{B}_1,\ \overset{+}{B}_2,\ \overset{+}{K}_1,\ \overset{+}{K}_2 \qquad \text{III-48}$$

## SECTION 7

## RECURSION PRODUCT FOR THE DESCENDING POLYNOMIAL SOLUTIONS

From III-6 we have

$$B_1 - A_1 = A_1(u-1) \qquad \text{III-49}$$

From III-4, using II-60, we have

$$B_2 K_1^* = A_1 \frac{G^2}{M} \qquad \text{III-50}$$

Also from III-4, using III-49, we have

$$B_2 K_1^* = A_1 (u-1) \frac{G}{N} \qquad \text{III-51}$$

Combining III-49 and III-50, we have

$$\left[(B_1 - A_1) - K_1^* B_2\right] = \frac{A_1}{M}\left[M(u-1) - G^2\right] \qquad \text{III-52}$$

Combining III-49 and III-51, we have

$$\left[(B_1 - A_1) - K_1^* B_2\right] = \frac{A_1}{N}\left[(N-G)(u-1)\right] \qquad \text{III-53}$$

Using III-1, III-5, III-49, III-52, III-53 in II-61, we have, after factoring $A_1$ and $B_2 j$ out of the numerator and denominator, respectively, as the recursion product for the descending series solutions

$$c^n_{k+\theta j} = \left(-\frac{A_1}{B_2 j}\right)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} \Big\{ j^2\phi^2 + 2j\phi(Mn^2 + Nn) + (Mn^2 + Nn)^2 + j(u-1)\phi + \left[M(u-1) - G^2\right]n^2 + \left[(N-G)(u-1)\right] n \Big\} \; c^n_k$$

$$= \left(-\frac{A_1}{B_2 j}\right)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} \Big\{ j^2\phi^2 + 2jN\phi n + 2jM\phi n^2 + \left[N^2 + M(u-1) - G^2\right]n^2 + 2MNn^3 + M^2n^4 + (N-G)(u-1)n + j(u-1)\phi \Big\} \; c^n_k \qquad \text{III-54}$$

Now in the discussion of closure of solutions following I-37, we let $\theta(n)$ be the value of $\theta$ for which we obtained the last non-vanishing coefficient $c^n_{k+\theta j}$ in the descending series solution, and showed that then the numerator of the recursion product I-30 had to be zero for $\phi_{max} = \theta(n)$.

Accordingly, the numerator of the recursion product III-54 must be zero for $\theta$ having the value $\theta(n)$, which we have previously shown is related to the order $k(n)$ and subdegree $h(n)$ by the equation

$$h(n) = k(n) + \theta(n)j \qquad \text{I-39"}$$

from which we have

$$\theta(n) = \frac{h(n) - k(n)}{j}$$

Using I-48 and III-5 in this, we have

$$\theta(n) = -\frac{M}{j}n^2 - \frac{N-G}{j}n \qquad \text{III-55}$$

By III-10 and III-12 this becomes

$$\theta(n) = an^2 + bn \qquad \text{III-56}$$

Since, as we have said, the numerator of the recursion product III-54 is zero for $\theta = \theta(n)$, then

$$\phi - \theta(n)$$

must be a factor of it. Using III-55 we have this factor as

$$\phi - \left[ - \frac{M}{j} n^2 - \frac{N-G}{j} n \right]$$

Multiplying by the constant j, we take the factor in the form

$$j\phi + (N-G)n + Mn^2 \qquad \text{III-57}$$

To obtain the other factor of the numerator of III-54, we divide this numerator by the factor III-57 as follows:

$$\begin{array}{r|llllllll}
 & j\phi & +\ (N+G)n & +\ Mn^2 & +\ (u-1) & & & & \\
\hline
j\phi + (N-G)n + Mn^2 & j^2\phi^2 & +\ 2jN\phi n & +\ 2jM\phi n^2 & +\left[N^2 + M(u-1) - G^2\right]n^2 & +\ 2MN n^3 & +\ M^2 n^4 & +\ (N-G)(u-1)n & +\ j(u-1)\phi \\
 & \underline{j^2\phi^2} & \underline{+\ j(N-G)\phi n} & \underline{+\ jM\phi n^2} & & & & & \\
 & & j(N+G)\phi n & +\ jM\phi n^2 & & & & & \\
 & & \underline{j(N+G)\phi n} & \underline{\phantom{+\ jM\phi n^2}} & \underline{+\ (N^2-G^2)\ n^2} & \underline{+\ M(N+G)n^3} & & & \\
 & & & jM\phi n^2 & +\ M(u-1)\ n^2 & +\ M(N-G)n^3 & & & \\
 & & & \underline{jM\phi n^2} & \underline{\phantom{+\ M(u-1)\ n^2}} & \underline{+\ M(N-G)n^3} & \underline{+\ M^2n^4} & & \\
 & & & & M(u-1)\ n^2 & & & & \\
 & & & & \underline{M(u-1)\ n^2} & \underline{\phantom{+\ M(N-G)n^3}} & \underline{\phantom{+\ M^2n^4}} & \underline{+\ (N-G)(u-1)n} & \underline{+\ j(u-1)\phi} \\
 & & & & & & 0 & &
\end{array}$$

Using III-57 and III-58 in III-54, and this result, together with III-56, in I-26, we have the Non-Classical Orthogonal Polynomial Solutions

$$y_n = c_k^n \sum_{\theta=0}^{an^2+bn} \left(-\frac{A_1}{B_2 j}\right)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} \left[Mn^2 + (N-G)n + j\phi\right] \cdot \left[Mn^2 + (N+G)n + (u-1) + j\phi\right] x^{Mn^2+Nn+\theta j} \qquad \text{III-59}$$

For the particular value $j = -1$, this becomes

$$y_n = c_k^n \sum_{\theta=0}^{an^2+bn} \left(\frac{A_1}{B_2}\right)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} \left[Mn^2 + (N-G)n - \phi\right] \cdot \left[Mn^2 + (N+G)n + (u-1) - \phi\right] x^{Mn^2+Nn-\theta} \qquad \text{III-60}$$

## SECTION 8

## DERIVATIVE DEFINITION OF THE POLYNOMIAL SOLUTIONS

We now consider the functions $Y_n$ given by the derivative definition

$$Y_r = e^{-Tx} D^R (x^S e^{Tx}) \qquad \text{III-61}$$

where D is the derivative operator; R,S,T constants; R a positive integer; $R < S$.

By the so-called "shifting formula" of differential equations, we have

$$Y_n = e^{-Tx} e^{Tx} (T + D)^R x^S$$

Expanding the binomial gives us

$$Y_n = \sum_{\theta=0}^{R} \binom{R}{\theta} T^{R-\theta} D^{\theta} x^S$$

Carrying out the indicated differentiation, we have, using II-74,

$$Y_n = T^R \sum_{\theta=0}^{R} \left(\frac{1}{T}\right)^{\theta} \binom{R}{\theta} \frac{\Gamma(S+1)}{\Gamma(S-\theta+1)} x^{S-\theta} \qquad \text{III-62}$$

Writing this out gives us

$$Y_n = T^R x^S + T^{R-1} \frac{R}{1!} S x^{S-1} + T^{R-2} \frac{R(R-1)}{2!} S(S-1) x^{S-2}$$

$$+ T^{R-3} \frac{R(R-1)(R-2)}{3!} S(S-1)(S-2) x^{S-3} + \dots$$

$$+ \frac{\Gamma(S+1)}{\Gamma(S-R+1)} x^{S-R} \qquad \text{III-63}$$

Accordingly, we may write

$$Y_n = T^R \sum_{\theta=0}^{R} \left(\frac{1}{T}\right)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (R-\phi)(S-\phi)\, x^{S-\theta} \qquad \text{III-64}$$

wherein as before following I-30 we define

$$\prod_{0}^{-1} = 1$$

Comparing III-60 with III-64, we see that in order to be able to identify the functions $Y_n$ given by III-64, which we obtained from the derivative definition III-61, with the solutions $y_n$ given by III-59, we must take $j = -1$. Then in III-60 and III-64 we make the following identifications:

$$\frac{1}{T} = \frac{A_1}{B_2} \qquad \text{III-65}$$

Using III-6 in this gives us

$$T = v \qquad \text{III-66}$$

We also have

$$\begin{array}{ll|rr} R & = Mn^2 + (N-G)n & -1 & \text{III-67} \\ S & = Mn^2 + (N+G)n + (u-1) & +1 & \text{III-68} \\ \hline \end{array}$$

$$S - R = 2Gn + u - 1 \qquad \text{III-69}$$

$$c^n_k = T^R \qquad \text{III-70}$$

By III-5, III-66, III-67 this becomes

$$c^n_{Mn^2+Nn} = v^{Mn^2+(N-G)n} \qquad \text{III-71}$$

From the last term in III-63 we have

$$c^{n}_{h} = \frac{\Gamma(S+1)}{\Gamma(S-R+1)} \qquad \text{III-72}$$

By I-48, III-68, III-69 this becomes

$$c^{n}_{Gn} = \frac{\Gamma\left[Mn^{2} + (N+G)n + u\right]}{\Gamma\left[2Gn + u\right]} \qquad \text{III-73}$$

With the preceding identification of the constants in III-60 and III-64, we now have

$$y_{n} = x^{H}\, Y_{n} \qquad \text{II-93'}$$

Equating the orders of both members of this equation gives us

$$k(n) = H + S$$

By III-5 and III-68 this becomes

$$Mn^{2} + Nn = H + Mn^{2} + (N+G)n + (u-1)$$

so that H is given by

$$H = -Gn - u + 1 \qquad \text{II-94"}$$

This same result is also obtainable by equating, with the aid of the last term in III-63, the subdegrees of both members of II-93' above

$$h(r) = H + (S-R)$$

Using I-48 and III-69, this becomes

$$Gn = H + 2Gn + u - 1$$

so that H is again given by

$$H = -Gn - u + 1 \qquad \text{II-94"'}$$

Using III-66, III-67, III-68 in III-61, and the result, together with II-94", in II-93', we have as the Derivative Definition of our Non-Classical Orthogonal Polynomial Solutions

$$y_n = x^{-Gn-u+1} e^{-vx} D^{Mn^2+(N-G)n} \left[ x^{Mn^2+(N+G)n+(u-1)} e^{vx} \right] \qquad \text{III-74}$$

## SECTION 9

## NON-VANISHING OF THE INTEGRATED SQUARE WHEN $n = 0$

Taking $n = o$ in III-74, we have

$$y_o = 1 \qquad \text{II-97'}$$

Using this and $j = -1$ in III-38, and taking the positive interval III-34, wherein $v < o < u$, we have for the integrated square of $y_o$

$$I(o) = \frac{1}{A_1}\int_o^{\infty} x^{u-2} e^{vx} (K_1 + K_2 x)\, dx \qquad \text{III-75}$$

For this integral to converge at the lower limit, III-7" (following III-38) required only that $u > 1$. Hence $(u-2)$ may be a fraction. In this case, for $I(o)$ to be real, we must have $x > o$. Then by III-34 the upper limit can only be $+\infty$, and III-35, III-47, III-48 cannot obtain.

Let

$$v = -J, \quad J > 0 \qquad \text{III-76}$$

Then III-75 becomes

$$I(o) = \frac{1}{A_1}\left[K_1 \int_0^{\infty} x^{u-2} e^{-Jx} dx + K_2 \int_0^{\infty} x^{u-1} e^{-Jx} dx\right] \qquad \text{III-77}$$

But the Gamma function is (12, p. 165, equation 11)

$$\Gamma(n) = J^n \int_0^{\infty} x^{n-1} e^{-Jx} dx, \quad J > 0 \qquad \text{III-78}$$

Using this, III-77 becomes

$$I(o) = \frac{1}{A_1}\left[K_1 \frac{\Gamma(u-1)}{J^{u-1}} + K_2 \frac{\Gamma(u)}{J^u}\right]$$

Using III-76 we may write this as

$$I(o) = \frac{1}{A_1} \left(-\frac{1}{v}\right)^u \Gamma(u-1) \left[K_2(u-1) - K_1(v)\right]$$

But from III-51 we have, with the aid of II-60 and III-6,

$$K_1 v = K_2(u-1) \frac{G}{N} \qquad \text{III-79}$$

Using this we may write our previous result as

$$I(o) = \frac{1}{A_1} (-v)^{-u} \Gamma(u-1) \left[ K_2(u-1) - K_2(u-1) \frac{G}{N} \right]$$

$$= (-1)^u \frac{K_2}{A_1} v^{-u} \Gamma(u) \left[ \frac{N-G}{N} \right] \qquad \text{III-80}$$

which proves III-14. Then by III-12 we have the proof of III-13

$$N \neq G \qquad \text{III-13'}$$

## SECTION 10

## INTEGRATED SQUARE FORMULA

Using III-37, with $j = -1$, and III-74 in I-25, with $m = n$, we have for the integrated square of $y_n$, taking the interval $(0, \infty)$ of III-34, wherein $v < o < u$,

$$I(n) = \frac{1}{A_1} \int_0^\infty x^{u-2} e^{vx} (K_1 + K_2 x) \; y_n \; x^{-Gn-u+1} e^{-vx}$$

$$\cdot \, D^{Mn^2+(N-G)n} \left[ x^{Mn^2+(N+G)n+(u-1)} e^{vx} \right] dx \qquad \text{III-81}$$

Let

$$\phi = x^{Mn^2+(N+G)n+(u-1)}\,e^{vx} \qquad \text{III-82}$$

and denote the derivatives of $\phi$ by superscripts in braces. Then II-81 becomes

$$I(n) = \frac{1}{A_1}\left[K_1 I_1(n) + K_2 I_2(n)\right] \qquad \text{III-83}$$

where

$$I_1(n) = \int_0^\infty (x^{-Gn-1}\,y_n)\,\phi^{\{Mn^2+(N-G)n\}}\,dx \qquad \text{III-84}$$

and

$$I_2(n) = \int_0^\infty (x^{-Gn}\,y_n)\,\phi^{\{Mn^2+(N-G)n\}}\,dx \qquad \text{III-85}$$

It is convenient to consider $I_2(n)$ first. Let

$$P_{Mn^2+(N-G)n,0} = x^{-Gn}\,y_n \qquad \text{III-86}$$

where P is a polynomial of order and subdegree as indicated by the respective subscripts. For $y_n$ is of order $k(n)$ and subdegree $h(n)$. Hence

$$P_{Mn^2+(N-G)n,0} = x^{-Gn}\left[c^n_{k(n)}\,x^{k(n)} + \ldots + c^n_{h(n)}\,x^{h(n)}\right]$$

Using the values of $k(n)$ and $h(n)$ as given by III-5 and I-48, respectively, we finally obtain

$$P_{Mn^2+(N-G)n,0} = x^{-Gn}\left(c^n_{Mn^2+Nn}\,x^{Mn^2+Nn} + \ldots + c^n_{Gn}\,x^{Gn}\right)$$

$$= c^n_{Mn^2+Nn}\,x^{Mn^2+(N-G)n} + \ldots + c^n_{Gn}\,x^0 \qquad \text{III-87}$$

Using III-86 in III-85 gives us

$$I_2(n) = \int_0^{\infty} P_{Mn^2+(N-G)n,0}\ \phi^{\left\{Mn^2+(N-G)n\right\}}\,dx$$

By parts

$$U = P_{Mn^2+(N-G)n,0} \qquad dV = \phi^{\{Mn^2+(N-G)n\}}\,dx$$

$$dU = P'_{Mn^2+(N-G)n,0}\,dx \qquad V = \phi^{\{Mn^2+(N-G)n-1\}}$$

Then

$$I_2(n) = \Big[P_{Mn^2+(N-G)n,0}\ \phi^{\{Mn^2+(N-G)n-1\}}\Big]_0^\infty$$

$$- \int_0^\infty P'_{Mn^2+(N-G)n,0}\ \phi^{\{Mn^2+(N-G)n-1\}}\,dx \qquad \text{III-88}$$

Using the expression for $\phi$ given by III-82, and performing the differentiation indicated in the bracket of III-88, we have

$$\phi^{\{Mn^2+(N-G)n-1\}} = D^{Mn^2+(N-G)n-1}\Big[x^{Mn^2+(N+G)n+(u-1)}\ e^{vx}\Big]$$

Using the shifting formula of differential equations, this becomes

$$e^{vx}\,(v+D)^{Mn^2+(N-G)n-1}\,x^{Mn^2+(N+G)n+(u-1)}$$

Expanding the binomial, we have

$$e^{vx}\sum_{\theta=0}^{Mn^2+(N-G)n-1}\binom{Mn^2+(N-G)n-1}{\theta}\,v^{Mn^2+(N-G)n-1-\theta}$$

$$\cdot\, D^{\theta}\,x^{Mn^2+(N+G)n+(u-1)}$$

Using II-74, we have

$$e^{vx}\sum_{\theta=0}^{Mn^2+(N-G)n-1}\binom{Mn^2+(N-G)n-1}{\theta}\,v^{Mn^2+(N-G)n-1-\theta}$$

$$\cdot\,\frac{\Gamma\left[Mn^2+(N+G)n+u\right]}{\Gamma\left[Mn^2+(N+G)n+u-\theta\right]}\,x^{Mn^2+(N+G)n+(u-1)-\theta}$$

With the aid of III-71 and III-73, we may write this as

$$\left[\frac{c^{n}_{Mn^2+Nn}}{v}\,x^{Mn^2+(N+G)n+(u-1)}+\dots+\frac{c^{n}_{Gn}}{2Gn+u}\,x^{2Gn+u}\right]e^{vx} \qquad \text{III-89}$$

Denote this by

$$g_{Mn^2+(N-G)n-1,0}\; x^{2Gn+u}\; e^{vx} \qquad \text{III-90}$$

where g is a polynomial of order and subdegree as indicated by the respective subscripts.

Using III-90 in the bracket of III-88 gives us

$$I_2(n) = \Big[P_{Mn^2+(N-G)n,0}\;\; g_{Mn^2+(N-G)n-1,0}\;\; x^{2Gn+u}\;\; e^{vx}\Big]_0^{\infty}$$

$$- \int_0^{\infty} P'_{Mn^2+(N-G)n,0}\;\; \phi^{\{Mn^2+(N-G)n-1\}}\, dx \qquad \text{III-91}$$

We see that for all $n > 0$ (if $n = 0$, the integration by parts is meaningless), the bracket in II-91 contains x to at least the power $2+u > 3$, since we have from III-7 that $u > 1$, and we know that $G \geqslant 1$. Also, by III-34, we have taken $v < 0$. Hence, for all $n > 0$, the bracket in III-91 vanishes.

If then we integrate by parts in III-85 a total of $Mn^2+(N-G)n$ times, we have, as an extension of III-91,

$$I_2(n) = (-1)^{Mn^2+(N-G)n} \int_0^\infty P^{\{Mn^2+(N-G)n\}}_{Mn^2+(N-G)n,0} \, \phi \, dx$$

since each time we apply the method, the bracket such as we had in III-91 will be of the same order and subdegree as in III-91. By III-87 we have

$$I_2(n) = (-1)^{Mn^2+(N-G)n} \; c^{n}_{Mn^2+Nn} \; \left[Mn^2+(N-G)n\right]! \int_0^\infty \phi \, dx$$

Using III-71 and III-82 in this, we have, with the aid of III-76

$$I_2(n) = (-1)^{Mn^2+(N-G)n} \; v^{Mn^2+(N-G)n} \; \left[Mn^2+(N-G)n\right]!$$

$$\cdot \int_0^\infty x^{Mn^2+(N+G)n+(u-1)} \; e^{-Jx} \; dx$$

Using III-78, this becomes, after replacing J by (-v) from III-76,

$$I_2(n) = (-1)^{Mn^2+(N-G)n} \; v^{Mn^2+(N-G)n} \; \left[Mn^2+(N-G)n\right]!$$

$$\cdot \; \frac{\Gamma\left[Mn^2+(N+G)n+u\right]}{(-v)^{Mn^2+(N+G)n+u}}$$

which simplifies to

$$I_2(n) = (-1)^{u} \; v^{-2Gn-u} \; \Gamma\left[Mn^2+(N+G)n+u\right] \left[Mn^2+(N-G)n\right]! \qquad \text{III-92}$$

We now consider $I_1(n)$ as given by III-84. Let

$$Q_{Mn^2+(N-G)n-1,-1} = x^{-Gn-1} \; y_n \qquad \text{III-93}$$

where Q is an algebraic function of order and subdegree as indicated by the respective subscripts. For $y_n$ is of order $k(n)$ and subdegree $h(n)$. Hence

$$Q_{Mn^2+(N-G)n-1,-1} = x^{-Gn-1}\left[c^{n}_{k(n)} \, x^{k(n)} + \ldots + c^{n}_{h(n)} \, x^{h(n)}\right]$$

Using the values of k(n) and h(n) as given by III-5 and I-48, respectively, we finally obtain

$$Q_{Mn^2+(N-G)n-1,-1} = x^{-Gn-1}\left(c^{n}_{Mn^2+Nn}\, x^{Mn^2+Nn} + \ldots + c^{n}_{Gn}\, x^{Gn}\right)$$

$$= c^{n}_{Mn^2+Nn}\, x^{Mn^2+(N-G)n-1} + \ldots + c^{n}_{Gn}\, x^{-1} \qquad \text{III-94}$$

Using III-93 in III-84 gives us

$$I_1(n) = \int_0^{\infty} Q_{Mn^2+(N-G)n-1,-1}\; \phi^{\{Mn^2+(N-G)n\}}\, dx$$

By parts

$$U = Q_{Mn^2+(N-G)n-1,-1} \qquad dV = \phi^{\{Mn^2+(N-G)n\}}\, dx$$

$$dU = Q'_{Mn^2+(N-G)n-1,-1}\, dx \qquad V = \phi^{\{Mn^2+(N-G)n-1\}}$$

Then

$$I_1(n) = \left[ Q_{Mn^2+(N-G)n-1,-1} \; \phi^{\{Mn^2+(N-G)n-1\}} \right]_0^\infty$$

$$- \int_0^\infty Q'_{Mn^2+(N-G)n-1,-1} \; \phi^{\{Mn^2+(N-G)n-1\}} \, dx \qquad \text{III-95}$$

The differentiation to be performed on the $\phi$ in the bracket here is the same as that in the bracket of III-88. Hence the result is the same as III-90. Accordingly, inserting this for $\phi^{\{Mn^2+(N-G)n-1\}}$ in the bracket of III-95, we have

$$I_1(n) = \left[ Q_{Mn^2+(N-G)n-1,-1} \; g_{Mn^2+(N-G)n-1,0} \; x^{2Gn+u} \; e^{vx} \right]_0^\infty$$

$$- \int_0^\infty Q'_{Mn^2+(N-G)n-1,-1} \; \phi^{\{Mn^2+(N-G)n-1\}} \, dx \qquad \text{III-96}$$

We see that for all $n > o$ (if $n = o$, the integration by parts is meaningless), the bracket in III-96 contains $x$ to at least the power $2+u-1 = u+1 > 2$, since we have from III-7 that $u > 1$, and we know that $G \geq 1$. Also, by III-34, we have taken $v < o$. Hence, for all $n > o$, the bracket in III-96 vanishes.

If then we integrate by parts in III-84 a total of $Mn^2+(N-G)n$ times, we have, as an extension of III-96,

$$I_1(n) = (-1)^{Mn^2+(N-G)n} \int_0^\infty Q^{\{Mn^2+(N-G)n\}}_{Mn^2+(N-G)n-1,-1} \phi \, dx$$

since each time we apply the method, the polynomial in the bracket such as we had in III-96, will be of the same order and subdegree as in III-96. By III-94 we then have

$$I_1(n) = (-1)^{Mn^2+(N-G)n} \, c^{n}_{Gn} \, (-1)^{Mn^2+(N-G)n} \left[Mn^2+(N-G)n\right]!$$

$$\cdot \int_0^\infty x^{-1-Mn^2-(N-G)n} \phi \, dx$$

Using III-73 and III-82, we have

$$I_1(n) = (-1)^{2\left[Mn^2+(N-G)n\right]} \frac{\Gamma\left[Mn^2+(N+G)n+u\right]}{\Gamma(2Gn+u)} \left[Mn^2+(N-G)n\right]!$$

$$\cdot \int_0^\infty x^{Mn^2+(N+G)n+(u-1)-1-Mn^2-(N-G)n} e^{vx} \, dx$$

With the aid of III-76, we have

$$I_1(n) = \frac{\Gamma\left[Mn^2+(N+G)n+u\right]}{\Gamma(2Gn+u)}\left[Mn^2+(N-G)n\right]!\int_0^\infty x^{2Gn+u-2}\,e^{-Jx}\,dx$$

Using III-78, and replacing J by (-v) from III-76, this becomes

$$I_1(n) = \frac{\Gamma\left[Mn^2+(N+G)n+u\right]}{\Gamma(2Gn+u)}\left[Mn^2+(N-G)n\right]!\;\frac{\Gamma(2Gn+u-1)}{(-v)^{2Gn+u-1}}$$

$$= (-1)^{u-1}\,v^{-2Gn-u+1}\,\frac{\Gamma\left[Mn^2+(N+G)n+u\right]}{2Gn+u-1}\left[Mn^2+(N-G)n\right]! \qquad \text{III-97}$$

Using III-92 and III-97 in III-83, we have

$$I(n) = \frac{1}{A_1}\Big[(-1)^{u}\,K_2\,v^{-2Gn-u}\,\Gamma\left[Mn^2+(N+G)n+u\right]\left[Mn^2+(N-G)n\right]!$$

$$+\,(-1)^{u-1}\,K_1\,v^{-2Gn-u+1}\,\frac{\Gamma\left[Mn^2+(N+G)n+u\right]}{2Gn+u-1}\left[Mn^2+(N-G)n\right]!\Big] \qquad \text{III-98}$$

$$I(n) = \frac{1}{A_1} (-1)^u v^{-2Gn-u} \Gamma\left[Mn^2+(N+G)n+u\right] \left[Mn^2+(N-G)n\right]!$$

$$\cdot \left[\frac{K_2(2Gn+u-1) - K_1 v}{2Gn+u-1}\right] \qquad \text{III-99}$$

But equating the right members of III-50 and III-51, and dividing out an $A_1G$ gives us

$$\frac{G}{M} = \frac{u-1}{N}$$

from which

$$u-1 = \frac{GN}{M} \qquad \text{III-100}$$

Using II-60 and III-6 in III-50, and multiplying the result by $K_2/A_1$ , we have

$$K_1 v = \frac{K_2 G^2}{M} \qquad \text{III-101}$$

Using III-100 and III-101 in the bracket of III-99, we have

$$\left[\frac{K_2\left(2Gn + \frac{GN}{M}\right) - \frac{K_2G^2}{M}}{2Gn + \frac{GN}{M}}\right]$$

Multiplying numerator and denominator by $\frac{M}{G}$, and using the result in place of the bracket of III-99, we have as the integrated square formula

$$I(n) = (-1)^u \frac{K_2}{A_1} v^{-2Gn-u} \left[\frac{2Mn+(N-G)}{2Mn+N}\right]$$

$$\cdot \Gamma\left[Mn^2+(N+G)n+u\right] \left[Mn^2+(N-G)n\right]! \qquad \text{III-102}$$

Using III-101 in the form

$$K_2 = K_1 v \frac{M}{G^2}$$

in III-102, we have as an alternate form of the integrated square formula

$$I(n) = (-1)^{u} \frac{K_1}{A_1} v^{-2Gn-u+1} \frac{M}{G^2} \left[\frac{2Mn+(N-G)}{2Mn+N}\right]$$

$$\cdot \Gamma\left[Mn^2+(N+G)n+u\right] \left[Mn^2+(N-G)n\right]! \qquad \text{III-103}$$

For $n = o$, III-102 becomes

$$I(o) = (-1)^{u} \frac{K_2}{A_1} v^{-u} \Gamma(u) \left[\frac{N-G}{N}\right] \qquad \text{III-80'}$$

as we had before by the direct method for $I(o)$.

Accordingly, we conclude that the formulas III-102 and III-103 for the integrated square are valid for all $n$, including $n = o$.

## SECTION 11

## NON-TELESCOPING OF THE RECURSION PRODUCT OF THE POLYNOMIAL SOLUTIONS

Using III-1 in II-132 and II-137 gives $A_1 = 0$, since we dismiss $j = o$ as giving monomials. But then the differential equation I-1 is degenerated to first order with the result that the orthogonality property is annihilated. Thus there is no telescoping which preserves the orthogonality property.

## SECTION 12

## SUMMARY OF RESULTS

$(A_1x^2)y_n'' + (B_1x + B_2x^{-j+1})y_n' + L(n)\left[K_1 + K_2x^{-j}\right]y_n = 0$ I-1 I-22 I-27 III-1

$u = \dfrac{B_1}{A_1} > 1$ ; $v = \dfrac{B_2}{A_1} < 0$ , and $K_2^* = \dfrac{K_2}{K_1} < 0$ if u is III-6 III-7 III-34 I-50 III-75+

fractional. $A_2 = 0$. $\overset{+}{A}_1, \overset{+}{B}_1, \overset{-}{B}_2, \overset{\mp}{K}_1, \overset{\pm}{K}_2$, for the interval III-1 III-45 III-46

$(0,\infty)$, $L(n) \gtrless 0$ . $\dfrac{N}{G} = \dfrac{K_2}{v}(u-1)$ , $M = \dfrac{K_2^*}{v}G^2$ , $N \neq G$ III-4 III-6 I-50 III-14

$k(n) = Mn^2 + Nn$, $h(n) = Gn$, $G = \left|\dfrac{B_2K_1j}{Q}\right|$ III-5 I-48 III-28

$Q =$ g.c.d. $\sqrt{|B_2K_1Hj|}$, $J$, $B_2j\sqrt{|K_1E|}$, $B_2Fj$ III-27

$J =$ g.c.d. $B_2K_1j$, $\begin{vmatrix} K_1 & K_2 & 0 \\ B_1 & B_2 & 1 \\ A_1 & 0 & 1 \end{vmatrix}$. $H =$ g.c.d. $A_1K_2$, $B_2K_1j$ III-24 III-21

$E =$ g.c.d. $A_1$, $K_1$ $F =$ g.c.d. $(B_1 - A_1)$, $K_1$ II-27' II-30'

$$L(n) = -\frac{B_2}{K_2}\,k(n) = \frac{-A_1 h^2(n) - (B_1 - A_1)\,h(n)}{K_1}$$ III-39 II-49'

$$W = \frac{1}{A_1}\,x^{u-2}\,e^{-\frac{v}{j}x^{-j}}\,(K_1 + K_2 x^{-j})$$ III-37

$$y_n = c_k^n \sum_{\theta=0}^{\frac{Mn^2+(N-G)n}{-j}} \left(-\frac{A_1}{B_2 j}\right)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} \left[Mn^2 + (N-G)n + j\phi\right]$$

$$\cdot \left[Mn^2 + (N+G)n + (u-1) + j\phi\right] x^{Mn^2+Nn+\theta j}$$ III-59 III-56 III-55

$$c_k^n = v^{Mn^2+(N-G)n}$$ When Derivative Definition is used, then $j = -1$ III-71 III-5

$$y_n = x^{-Gn-u+1}\,e^{-vx}\,D^{Mn^2+(N-G)n}\left[x^{Mn^2+(N+G)n+(u-1)}\,e^{vx}\right]$$ III-74

$$I(n) = (-1)^u\,\frac{K_2}{A_1}\,v^{-2Gn-u}\left[\frac{2Mn+(N-G)}{2Mn+N}\right]$$

$$\cdot\;\Gamma\left[Mn^2+(N+G)n+u\right]\left[Mn^2+(N-G)n\right]!$$ III-102

## SECTION 13

## NUMERICAL EXAMPLES

EXAMPLE 1 u,v integers; $N > G = 1$
constant of proportionality = 1

$A_1 = 1,\ A_2 = 0,\ B_1 = 3,\ B_2 = -1,\ K_1 = -1,\ K_2 = 1,\ j = -1$

$u = \frac{3}{1} = 3 > 1,\ v = \frac{-1}{1} = -1 < 0,\ K_2^* = \frac{1}{-1} = -1 < 0$. Thus

III-1,-7,-34,-45 all hold. $\frac{N}{G} = \frac{-1}{-1}(3-1) = \frac{2}{1}$, so that

taking $G = 1$, $N = 2$, we have $M = \frac{-1}{-1}(1)^2 = 1$,

$k(n) = n^2+2n$, $h(n) = n$, and III-14 holds. Also

$$L(n) = \frac{-(1)n^2 - (3-1)n}{-1} = n^2 + 2n = n(n+2)$$

Therefore the differential equation

$$(x^2)y_n'' + (3x - x^2)y_n' + n(n+2)(-1 + x)y_n = 0$$

has polynomial solutions

$$y_n = (-1)^{n^2+n} \sum_{\theta=0}^{n^2+n} (-1)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (n^2+n-\phi)$$

$$\cdot (n^2+3n+2-\phi)\, x^{n^2+2n-\theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = x e^{-x} (-1+x) = (x^2 - x) e^{-x}$$

having the derivative definition

$$y_n = x^{-n-2} e^{x} D^{n^2+n} \left[x^{n^2+3n+2} e^{-x}\right]$$

and integrated square formula

$$I(n) = (-1)^3 \frac{1}{1} (-1)^{-2n-3} \left[\frac{2n+1}{2n+2}\right] \Gamma(n^2+3n+3)\ (n^2+n)!$$

$$= \left[\frac{2n+1}{2n+2}\right] (n^2+3n+2)!\ \ (n^2+n)!$$

$y_0 = 1$

$$y_1 = x^{-3} e^{x} D^{2} (x^{6} e^{-x}) = x^{-3} (D-1)^{2} x^{6} = x^{-3} (D^{2} - 2D + 1) x^{6}$$

$$= x^{-3} (6 \cdot 5x^{4} - 2 \cdot 6x^{5} + x^{6}) = x^{3} - 12x^{2} + 30x$$

$$y_2 = x^{-4} e^{x} D^{6} (x^{12} e^{-x}) = x^{-4} (D-1)^{6} x^{12}$$

$$= x^{-4} (D^{6} - 6D^{5} + 15D^{4} - 20D^{3} + 15D^{2} - 6D + 1) x^{12}$$

$$= x^{-4} (12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7x^{6} - 6 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8x^{7} + 15 \cdot 12 \cdot 11 \cdot 10 \cdot 9x^{8}$$
$$- 20 \cdot 12 \cdot 11 \cdot 10x^{9} + 15 \cdot 12 \cdot 11x^{10} - 6 \cdot 12x^{11} + x^{12})$$

$$= x^{8} - 72x^{7} + 1{,}980x^{6} - 26{,}400x^{5} + 178{,}200x^{4}$$
$$- 570{,}240x^{3} + 665{,}280x^{2}$$

$$I(0) = \frac{1}{2} (2)!\ (0)! = 1$$

$$I(1) = \frac{3}{4} (6)!\ (2)! = 9\ (5)! = 9\ (120) = 1080$$

$$I(2) = \frac{5}{6} (12)!\ (6)! = 5\ (5)!\ (12)! = 5(120)(12)! = 600\ (12)!$$

$$= 600\ (479{,}001{,}600) = 287{,}400{,}960{,}000$$

EXAMPLE 2 $u, v$ fractions; $N > G = 1$
constant of proportionality $= 1$

$A_1 = 2$, $A_2 = 0$, $B_1 = 3$, $B_2 = -1$, $K_1 = -1$, $K_2 = 2$, $j = -1$

$u = \frac{3}{2} > 1$, $v = \frac{-1}{2} = -\frac{1}{2} < 0$, $K_2^* = \frac{2}{-1} = -2 < 0$. Thus

III-1, -7, -34, -45, -75ff. all hold. $\frac{N}{G} = \frac{-2}{-\frac{1}{2}}\left(\frac{3}{2} - 1\right) = \frac{2}{1}$,

so that taking $G = 1$, $N = 2$, we have $M = \frac{-2}{-\frac{1}{2}}(1)^2 = 4$,

$k(n) = 4n^2 + 2n$, $h(n) = n$, and III-14 holds. Also

$$L(n) = \frac{-(2)n^2 - (3-2)n}{-1} = 2n^2 + n = n(2n+1)$$

Therefore the differential equation

$$(2x^2)y_n'' + (3x - x^2)y_n' + n(2n+1)(-1 + 2x)y_n = 0$$

has polynomial solutions

$$y_n = \left(-\frac{1}{2}\right)^{4n^2+n} \sum_{\theta=0}^{4n^2+n} (-2)^{\theta} \, \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (4n^2+n-\phi)$$

$$\cdot \, (4n^2+3n+\frac{1}{2}-\phi) \; x^{4n^2+n-\theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = \frac{1}{2} \, x^{-1/2} \, e^{-x/2} \, (-1+2x) = (x^{1/2} - \frac{x^{-1/2}}{2}) \, e^{-x/2}$$

having the derivative definition

$$y_n = x^{-n-1/2} \, e^{x/2} \, D^{4n^2+n} \left[ x^{4n^2+3n+1/2} \, e^{-x/2} \right]$$

and integrated square formula

$$I(n) = (-1)^{3/2} \, \frac{2}{2} \left(-\frac{1}{2}\right)^{-2n-3/2} \left[\frac{8n+1}{8n+2}\right]$$

$$\Gamma(4n^2+3n+\frac{3}{2}) \; (4n^2+n)!$$

$$= 2^{2n+1/2} \left[\frac{8n+1}{4n+1}\right] \Gamma(4n^2+3n+\frac{3}{2}) \; (4n^2+n)!$$

EXAMPLE 3 $u, v$ integers; $N > G \neq 1$
constant of proportionality = 1

$A_1 = 1$, $A_2 = 0$, $B_1 = 2$, $B_2 = -2$, $K_1 = -1$, $K_2 = 3$, $j = -1$

$u = \frac{2}{1} = 2 > 1$, $v = \frac{-2}{1} = -2 < 0$, $K_2^* = \frac{3}{-1} = -3 < 0$. Thus III-1,-7,-34,-45 all hold. $\frac{N}{G} = \frac{-3}{-2}(2-1) = \frac{3}{2}$, so that taking $G = 2$, $N = 3$, we have $M = \frac{-3}{-2}(2)^2 = 6$, $k(n) = 6n^2 + 3n$, $h(n) = 2n$, and III-14 holds. Also

$$L(n) = \frac{-(1)(2n)^2 - (2-1)(2n)}{-1} = 4n^2 + 2n = 2n(2n+1)$$

Therefore the differential equation

$$(x^2)y_n'' + (2x - 2x^2)y_n' + 2n(2n+1)(-1 + 3x)y_n = 0$$

has polynomial solutions

$$y_n = (-2)^{6n^2+n} \sum_{\theta=0}^{6n^2+n} \left(-\frac{1}{2}\right)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (6n^2+n-\phi)$$

$$\cdot\, (6n^2+5n+1-\phi)\; x^{6n^2+3n-\theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = x^{0} e^{-2x} (-1+3x) = (3x-1)\, e^{-2x}$$

having the derivative definition

$$y_n = x^{-2n-1} e^{2x} D^{6n^2+n} \left[ x^{6n^2+5n+1} e^{-2x} \right]$$

and integrated square formula

$$I(n) = (-1)^{2}\, \frac{3}{1}\, (-2)^{-4n-2} \left[\frac{12n+1}{12n+3}\right]$$

$$\cdot\, \Gamma(6n^2+5n+2)\; (6n^2+n)!$$

$$= 2^{-4n-2} \left[\frac{12n+1}{4n+1}\right] (6n^2+5n+1)!\; (6n^2+n)!$$

EXAMPLE 4 u fractional; $N > G \neq 1$
constant of proportionality = 1

$A_1 = 2$, $A_2 = 0$, $B_1 = 3$, $B_2 = -2$, $K_1 = -1$, $K_2 = 3$, $j = -1$

$u = \frac{3}{2} > 1$, $v = \frac{-2}{2} = -1 < 0$, $K_2^* = \frac{3}{-1} = -3 < 0$. Thus

III-1,-7,-34,-45,-75ff. all hold. $\frac{N}{G} = \frac{-3}{-1}\left(\frac{3}{2} - 1\right) = \frac{3}{2}$,

so that taking $G = 2$, $N = 3$, we have $M = \frac{-3}{-1}(2)^2 = 12$,

$k(n) = 12n^2 + 3n$, $h(n) = 2n$, and III-14 holds. Also

$$L(n) = \frac{-(2)(2n)^2 - (3-2)(2n)}{-1} = 8n^2 + 2n = 2n(4n+1)$$

Therefore the differential equation

$$(2x^2)y_n'' + (3x - 2x^2)y_n' + 2n(4n+1)(-1 + 3x)y_n = 0$$

has polynomial solutions

$$y_n = (-1)^{12n^2+n} \sum_{\theta=0}^{12n^2+n} (-1)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (12n^2 + n - \phi)$$

$$\cdot \, (12n^2 + 5n + \frac{1}{2} - \phi) \; x^{12n^2+3n-\theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = \frac{1}{2} x^{-1/2} e^{-x} (-1 + 3x) = \frac{1}{2} (3x^{1/2} - x^{-1/2}) e^{-x}$$

having the derivative definition

$$y_n = x^{-2n-1/2} e^{x} D^{12n^2+n} \left[ x^{12n^2+5n+1/2} e^{-x} \right]$$

and integrated square formula

$$I(n) = (-1)^{3/2} \frac{3}{2} (-1)^{-4n-3/2} \left[ \frac{24n+1}{24n+3} \right]$$

$$\cdot \, \Gamma(12n^2 + 5n + \frac{3}{2}) \; (12n^2+n)!$$

$$= \frac{1}{2} \left[ \frac{24n+1}{8n+1} \right] \Gamma(12n^2+5n+\frac{3}{2}) \; (12n^2+n)!$$

EXAMPLE 5 u,v integers; $N < G \neq 1$
constant of proportionality = 1

$A_1 = 1, \; A_2 = 0, \; B_1 = 2, \; B_2 = -1, \; K_1 = -2, \; K_2 = 1, \; j = -1$

$u = \frac{2}{1} = 2 > 1, \; v = \frac{-1}{1} = -1 < 0, \; K_2^* = \frac{1}{-2} = -\frac{1}{2} < 0$ . Thus

III-1,-7,-34,-45 all hold. $\frac{N}{G} = \frac{-\frac{1}{2}}{-1}(2-1) = \frac{1}{2}$, so that

taking $G = 2$, $N = 1$, we have $M = \frac{-\frac{1}{2}}{-1}(2)^2 = 2$,

$k(n) = 2n^2 + n$, $h(n) = 2n$, and III-14 holds. Also

$$L(n) = \frac{-(1)(2n)^2 - (2-1)(2n)}{-2} = 2n^2 + n = n(2n+1)$$

Therefore the differential equation

$$(x^2)y_n'' + (2x - x^2)y_n' + n(2n+1)(-2 + x)y_n = 0$$

has polynomial solutions

$$y_n = (-1)^{2n^2-n} \sum_{\theta=0}^{2n^2-n} (-1)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (2n^2 - n - \phi)$$

$$\cdot (2n^2 + 3n + 1 - \phi) \, x^{2n^2+n-\theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = e^{-x} (-2 + x) = (x - 2) \, e^{-x}$$

having the derivative definition

$$y_n = x^{-2n-1} \, e^{x} \, D^{2n^2-n} \left[ x^{2n^2+3n+1} \, e^{-x} \right]$$

and integrated square formula

$$I(n) = (-1)^{2} \, \frac{1}{1} \, (-1)^{-4n-2} \left[\frac{4n-1}{4n+1}\right] \Gamma(2n^2 + 3n + 2) \, (2n^2 - n)!$$

$$= \left[\frac{4n-1}{4n+1}\right] (2n^2 + 3n + 1)! \, (2n^2 - n)!$$

EXAMPLE 6 u fractional; $N < G \neq 1$
constant of proportionality = 1

$A_1 = 2$, $A_2 = 0$, $B_1 = 3$, $B_2 = -2$, $K_1 = -1$, $K_2 = 1$, $j = -1$

$u = \frac{3}{2} > 1$, $v = \frac{-2}{2} = -1 < 0$, $K_2^* = \frac{1}{-1} = -1 < 0$. Thus

III-1,-7,-34,-45,-75ff. all hold. $\frac{N}{G} = \frac{-1}{-1}(\frac{3}{2} - 1) = \frac{1}{2}$,

so that taking $G = 2$, $N = 1$, we have $M = \frac{-1}{-1}(2)^2 = 4$,

$k(n) = 4n^2 + n$, $h(n) = 2n$, and III-14 holds. Also

$$L(n) = \frac{-(2)(2n)^2 - (3-2)(2n)}{-1} = 8n^2 + 2n = 2n(4n+1)$$

Therefore the differential equation

$$(2x^2)y_n'' + (3x - 2x^2)y_n' + 2n(4n+1)(-1 + x)y_n = 0$$

has polynomial solutions

$$y_n = (-1)^{4n^2-n} \sum_{\theta=0}^{4n^2-n} (-1)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (4n^2 - n - \phi)$$

$$\cdot (4n^2 + 3n + \frac{1}{2} - \phi) \; x^{4n^2+n-\theta}$$

orthogonal in $(0,\infty)$. with weight function

$$W = \frac{1}{2} x^{-1/2} e^{-x} (-1 + x) = \left(\frac{x^{1/2} - x^{-1/2}}{2}\right) e^{-x}$$

having the derivative definition

$$y_n = x^{-2n-1/2} e^{x} D^{4n^2-n} \left[x^{4n^2+3n+1/2} e^{-x}\right]$$

and integrated square formula

$$I(n) = (-1)^{3/2} \frac{1}{2} (-1)^{-4n-3/2} \left[\frac{8n-1}{8n+1}\right]$$

$$\Gamma(4n^2 + 3n + \frac{3}{2}) \; (4n^2 - n)!$$

$$= \frac{1}{2} \left[\frac{8n-1}{8n+1}\right] \Gamma(4n^2 + 3n + \frac{3}{2}) \; (4n^2 - n)!$$

EXAMPLE 7 u,v integers; $N > G \neq 1$
constant of proportionality $\neq 1$

$A_1 = 1, \quad A_2 = 0, \quad B_1 = 2, \quad B_2 = -1, \quad K_1 = -4, \quad K_2 = 6, \quad j = -1$

$u = \frac{2}{1} = 2 > 1, \quad v = \frac{-1}{1} = -1 < 0, \quad K_2^* = \frac{6}{-4} = -\frac{3}{2} < 0$ . Thus III-1,-7,-34,-45 all hold. $\frac{N}{G} = \frac{-\frac{3}{2}}{-1}(2-1) = \frac{3}{2}$ , so that if we try $G = 2$, $N = 3$, we have $M = \frac{-\frac{3}{2}}{-1}(2)^2 = 6$ ,

$k(n) = 6n^2 + 3n$, $h(n) = 2n$, and III-14 holds. But

$$L(n) = \frac{-(1)(2n)^2 - (2-1)(2n)}{-4} = \frac{2n^2 + n}{2}$$

which is not an integer. But by inspection of this result for L(n), we see that if we augment the above trial values of N and G by a constant of proportionality 2, we will have

$G = 4$, $N = 6$, $M = \frac{-\frac{3}{2}}{-1}(4)^2 = 24$, $k(n) = 24n^2 + 6n$, $h(n) = 4n$,

$$L(n) = \frac{-(1)(4n)^2 - (2-1)(4n)}{-4} = 4n^2 + n = n(4n+1)$$

an integer. Therefore the differential equation

$$(x^2)y_n'' + (2x - x^2)y_n' + n(4n+1)(-4 + 6x)y_n = 0$$

has polynomial solutions

$$y_n = (-1)^{24n^2+2n} \sum_{\theta=0}^{24n^2+2n} (-1)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (24n^2+2n-\phi)$$

$$\cdot (24n^2+10n+1-\phi)\, x^{24n^2+6n-\theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = x^{0} e^{-x} (-4+6x) = (6x-4)\, e^{-x}$$

having the derivative definition

$$y_n = x^{-4n-1} e^{x} D^{24n^2+2n} \left[ x^{24n^2+10n+1} e^{-x} \right]$$

and integrated square formula

$$I(n) = (-1)^{2} \frac{6}{1} (-1)^{-8n-2} \left[ \frac{48n+2}{48n+6} \right]$$

$$\cdot \Gamma(24n^2+10n+2)\,(24n^2+2n)!$$

$$= 2 \left[ \frac{24n+1}{8n+1} \right] (24n^2+10n+1)!\,(24n^2+2n)!$$

EXAMPLE 8 u,v fractions; $N > G \neq 1$
constant of proportionality $\neq 1$

$A_1 = 3$, $A_2 = 0$, $B_1 = 5$, $B_2 = -1$, $K_1 = -2$, $K_2 = 2$, $j = -1$

$u = \frac{5}{3} > 1$, $v = \frac{-1}{3} = -\frac{1}{3} < 0$, $K_2^* = \frac{2}{-2} = -1 < 0$. Thus

III-1,-7,-34,-45,-75ff. all hold. $\frac{N}{G} = \frac{-1}{-\frac{1}{3}}\left(\frac{5}{3} - 1\right) = \frac{2}{1}$,

so that if we try $G = 1$, $N = 2$, we have $M = \frac{-1}{-\frac{1}{3}}(1)^2 = 3$,

$k(n) = 3n^2 + 2n$, $h(n) = n$, and III-14 holds. But

$$L(n) = \frac{-(3)n^2 - (5-3)n}{-2} = \frac{3n^2 + 2n}{2}$$

which is not an integer. But by inspection of this result for $L(n)$, we see that if we augment the above trial values of N and G by a constant of proportionality 2, we will have

$G = 2$, $N = 4$, $M = \frac{-1}{-\frac{1}{3}}(2)^2 = 12$, $k(n) = 12n^2 + 4n$, $h(n) = 2n$,

$$L(n) = \frac{-(3)(2n)^2 - (5-3)(2n)}{-2} = 6n^2 + 2n = 2n(3n+1)$$

an integer. Therefore the differential equation

$$(3x^2)y_n'' + (5x - x^2)y_n' + 2n(3n+1)(-2 + 2x)y_n = 0$$

has polynomial solutions

$$y_n = \left(-\frac{1}{3}\right)^{12n^2+2n} \sum_{\theta=0}^{12n^2+2n} (-3)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (12n^2+2n-\phi) \cdot (12n^2+6n+\frac{2}{3}-\phi)\, x^{12n^2+4n-\theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = \frac{1}{3} x^{-1/3} e^{-x/3} (-2+2x) = \frac{2}{3} (x^{2/3} - x^{-1/3})\, e^{-x/3}$$

having the derivative definition

$$y_n = x^{-2n-2/3} e^{x/3} D^{12n^2+2n} \left[ x^{12n^2+6n+2/3} e^{-x/3} \right]$$

and integrated square formula

$$I(n) = (-1)^{5/3} \frac{2}{3} \left(-\frac{1}{3}\right)^{-4n-5/3} \left[\frac{24n+2}{24n+4}\right] \cdot \Gamma(12n^2+6n+\frac{5}{3})\, (12n^2+2n)!$$

$$= 3^{4n+2/3} \left[\frac{12n+1}{6n+1}\right] \Gamma(12n^2+6n+\frac{5}{3})\, (12n^2+2n)!$$

EXAMPLE 9 $u, v$ integers; $N < G \neq 1$
constant of proportionality $\neq 1$

$A_1 = 1$, $A_2 = 0$, $B_1 = 3$, $B_2 = -1$, $K_1 = -8$, $K_2 = 2$, $j = -1$

$u = \frac{3}{1} = 3 > 1$, $v = \frac{-1}{1} = -1 < 0$, $K^*_2 = \frac{2}{-8} = -\frac{1}{4} < 0$. Thus III-1,-7,-34,-45 all hold. $\frac{N}{G} = \frac{-\frac{1}{4}}{-1}(3-1) = \frac{1}{2}$, so that if we try $G = 2$, $N = 1$, we have $M = \frac{-\frac{1}{4}}{-1}(2)^2 = 1$, $k(n) = n^2 + n$, $h(n) = 2n$, and III-14 holds. But

$$L(n) = \frac{-(1)(2n)^2 - (3-1)(2n)}{-8} = \frac{n^2 + n}{2}$$

which is not an integer. But by inspection of this result for $L(n)$, we see that if we augment the above trial values of $N$ and $G$ by a constant of proportionality 2, we will have

$G = 4$, $N = 2$, $M = \frac{-\frac{1}{4}}{-1}(4)^2 = 4$, $k(n) = 4n^2 + 2n$, $h(n) = 4n$,

$$L(n) = \frac{-(1)(4n)^2 - (3-1)(4n)}{-8} = 2n^2 + n = n(2n+1)$$

an integer. Therefore the differential equation

$$(x^2)y''_n + (3x - x^2)y'_n + n(2n+1)(-8 + 2x)y_n = 0$$

has polynomial solutions

$$y_n = (-1)^{4n^2-2n} \sum_{\theta=0}^{4n^2-2n} (-1)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (4n^2 - 2n - \phi)$$

$$\cdot (4n^2 + 6n + 2 - \phi)\, x^{4n^2 + 2n - \theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = x\, e^{-x} (-8 + 2x) = 2x(x-4)\, e^{-x}$$

having the derivative definition

$$y_n = x^{-4n-2}\, e^{x}\, D^{4n^2-2n} \left[ x^{4n^2+6n+2}\, e^{-x} \right]$$

and integrated square formula

$$I(n) = (-1)^{3} \frac{2}{1} (-1)^{-8n-3} \left[\frac{8n-2}{8n+2}\right]$$

$$\cdot\, \Gamma(4n^2 + 6n + 3)\, (4n^2 - 2n)!$$

$$= 2 \left[\frac{4n-1}{4n+1}\right] (4n^2 + 6n + 2)!\, (4n^2 - 2n)!$$

EXAMPLE 10 u fractional; $N < G \neq 1$
constant of proportionality $\neq 1$

$A_1 = 2$, $A_2 = 0$, $B_1 = 3$, $B_2 = -2$, $K_1 = -3$, $K_2 = 3$, $j = -1$

$u = \frac{3}{2} > 1$, $v = \frac{-2}{2} = -1 < 0$, $K_2^* = \frac{3}{-3} = -1 < 0$. Thus

III-1,-7,-34,-45,-75ff. all hold. $\frac{N}{G} = \frac{-1}{-1}\left(\frac{3}{2} - 1\right) = \frac{1}{2}$,

so that if we try $G = 2$, $N = 1$, we have $M = \frac{-1}{-1}(2)^2 = 4$,

$k(n) = 4n^2 + n$, $h(n) = 2n$, and III-14 holds. But

$$L(n) = \frac{-(2)(2n)^2 - (3-2)(2n)}{-3} = \frac{8n^2 + 2n}{3}$$

which is not an integer. But by inspection of this result for L(n), we see that if we augment the above trial values of N and G by a constant of proportionality 3, we will have

$G = 6$, $N = 3$, $M = \frac{-1}{-1}(6)^2 = 36$, $k(n) = 36n^2 + 3n$, $h(n) = 6n$,

$$L(n) = \frac{-(2)(6n)^2 - (3-2)(6n)}{-3} = 24n^2 + 2n = 2n(12n+1)$$

an integer. Therefore the differential equation

$$(2x^2)y_n'' + (3x - 2x^2)y_n' + 2n(12n+1)(-3 + 3x)y_n = 0$$

has polynomial solutions

$$y_n = (-1)^{36n^2-3n} \sum_{\theta=0}^{36n^2-3n} (-1)^{\theta} \frac{1}{\theta!} \prod_{\phi=0}^{\theta-1} (36n^2-3n-\phi)$$

$$\cdot (36n^2+9n+\frac{1}{2}-\phi)\; x^{36n^2+3n-\theta}$$

orthogonal in $(0,\infty)$, with weight function

$$W = \frac{1}{2} x^{-1/2} e^{-x} (-3+3x) = \frac{3}{2} (x^{1/2} - x^{-1/2}) e^{-x}$$

having the derivative definition

$$y_n = x^{-6n-1/2} e^{x} D^{36n^2-3n} \left[ x^{36n^2+9n+1/2} e^{-x} \right]$$

and integrated square formula

$$I(n) = (-1)^{3/2} \frac{3}{2} (-1)^{-12n-3/2} \left[ \frac{72n-3}{72n+3} \right]$$

$$\cdot \Gamma(36n^2+9n+\frac{3}{2})\; (36n^2-3n)!$$

$$= \frac{3}{2} \left[ \frac{24n-1}{24n+1} \right] \Gamma(36n^2+9n+\frac{3}{2})\; (36n^2-3n)!$$

# PART IV

# PRACTICAL APPLICATION
of
THE RESULTS

## PRACTICAL APPLICATION OF THE RESULTS

We are now in a position to show the connection of our problem with a type of boundary value problem in mathematical physics.

Suppose we have the second order linear homogeneous partial differential equation

$$g\left(\frac{\partial^2 T}{\partial z^2}, \frac{\partial^2 T}{\partial x^2}, \frac{\partial T}{\partial z}, \frac{\partial T}{\partial x}, T, x, z\right) = 0 \qquad \text{IV-1}$$

Assume the solution of the form

$$T(x,z) = y(x)\, Z(z) \qquad \text{IV-2}$$

Inserting appropriate derivatives of IV-2 in IV-1 and separating variables, if this is possible (otherwise the method fails), we have

$$g_1(y'',y',y,x) = g_2(Z'',Z',Z,z) = L(n) \qquad \text{IV-3}$$

where L(n) is an assignable constant.

Suppose that $g_1 = L(n)$ can be resolved into the form of the differential equation considered in this paper, namely

$$py_n'' + qy_n' + t_n y_n = 0 \qquad \text{I-6'}$$

where p, q, $t_n$ are given by I-2, I-3, I-4, respectively.

For case I, of I-16, where $L_1$ is a constant independent of n, we may assign the order k(n) of the descending series solution as an integral function of an integer n, say k(n) = Nn, N an integer. Then the indicial equation I-41 determines the form required for $L_2(n)$; and for $L_1 = 0$, $j = -1$, or for $L_1 = 0$, $B_1 = 0$, $j = -2$, we will have an infinite set of orthogonal polynomial solutions $y_n(x)$, as indicated in the discussion of this case under closure of solutions, descending series, following I-42.

Or, if we assign the order h(n) of the ascending series solutions as an integral function of an integer n, say h(n) = Nn, N an integer, then the closure equation I-57' determines the form required for $L_2(n)$; and for $L_1 = 0$, $j = 1$, or for $L_1 = 0$, $B_1 = 0$, $j = 2$, we will again have an infinite set of orthogonal polynomial solutions $y_n(x)$, as indicated in the discussion of this case under closure of solutions, ascending series, following I-58.

For case II, of I-19, where $L_2$ is a constant independent of n, if we assign the subdegree h(n) of the descending series solutions as an integral function of an integer n, say h(n) = Gn, G an integer, then we will have either monomial solutions or a finite set of orthogonal polynomial solutions, as shown following I-54, neither of which suits our purposes here.

An argument similar to that following I-53' disposes of the descending series solutions, h merely replacing k.

For case III, of I-22, where $L_1(n) = K_1 L(n)$, and $L_2(n) = K_2 L(n)$, we may assign the subdegree $h(n)$ of the descending series solutions as $h(n) = Gn$, as in I-48, G being determined by II-34 or by III-28, according as II-1 or III-1 holds. Then the closure equation I-49 determines the form of $L(n)$, and we will have an infinite set of orthogonal polynomial solutions of one or the other of two new types, of order $k(n)$ as given by II-11 or III-3, according as II-1 or III-1 holds. (If neither of these conditions holds, we cannot have polynomial solutions, as shown following I-45.)

Or, if we assign the subdegree $k(n)$ of the ascending series solutions as $k(n) = Gn$, as in I-62, we have similar results, as indicated following I-62.

Thus from the solutions of $g_1 = L(n)$, which will be of the form

$$y(x) = E_1\, y_L(x) + \overline{E}_1\, \overline{y}_L(x) \qquad \text{IV-4}$$

with $E_1$, $\overline{E}_1$ constants, since we have supposed $g_1 = L(n)$

to be of the form I-6, namely second order linear, we may in cases I and III determine L(n) so as to obtain an infinite set of orthogonal polynomial solutions $y_n(x)$.

With such determination of L(n) the solutions of $g_2 = L(n)$ are then of the form

$$Z(z) = E_2 Z_n(z) + \bar{E}_2 \bar{Z}_n(z) \qquad \text{IV-5}$$

with $E_2$, $\bar{E}_2$ constants, if we further suppose that $g_2 = L(n)$ is also second order linear. Accordingly, IV-2 is now of the form

$$T_n(x,z) = y_n(x) \left[ E_2 Z_n(z) + \bar{E}_2 \bar{Z}_n(z) \right] \qquad \text{IV-6}$$

Now if boundary conditions, considerations of continuity, or convergence of solutions demands that say $\bar{E}_2 = 0$, then the solutions of IV-1 become

$$T_n(x,z) = C_n y_n(x) Z_n(z) \qquad \text{IV-7}$$

Furthermore, due to the supposed linear character of the given partial differential equation IV-1,

$$T(x,z) = \sum_{n=0}^{\infty} C_n Z_n(z) y_n(x) \qquad \text{IV-8}$$

is also a solution.

Finally, suppose that for $z = z_0$ we have the non-homogeneous boundary condition

$$T(x,z_0) = \sum_{n=0}^{\infty} C_n Z_n(z_0)\, y_n(x) = f(x) \qquad \text{IV-9}$$

If we multiply by $W\, y_m(x)\, dx$ and integrate from $x_0$ to $x_1$, we have

$$\int_{x_0}^{x_1} W\, y_m(x)\, f(x)\, dx = \int_{x_0}^{x_1} \sum_{n=0}^{\infty} C_n Z_n(z_0)\, W\, y_m(x)\, y_n(x)\, dx$$

$$= C_m Z_m(z_0) \int_{x_0}^{x_1} W\, y_m^2(x)\, dx \qquad \text{IV-10}$$

by the orthogonality property. Thus we have

$$C_n = \frac{\int_{x_0}^{x_1} W\, y_n(x)\, f(x)\, dx}{Z_n(z_0)\, I(n)} \qquad \text{IV-11}$$

where I(n) is given by II-128 or II-129, or by III-102 or III-103, according as II-1 or III-1 holds.

Using IV-11 in IV-8 gives the formal solution to the problem:

$$T(x,z) = \sum_{n=0}^{\infty} \frac{\int_{x_0}^{x_1} W\, y_n(x)\, f(x)\, dx}{Z_n(z_0)\, I(n)}\, Z_n(z)\, y_n(x) \qquad \text{IV-12}$$

# APPENDIX

## NOTATION

| SYMBOL | MEANING | INTRODUCED |
|---|---|---|
| $A_1$ | constant in the differential equation | I-1 |
| $A_2$ | constant in the differential equation | I-1 |
| $B_1$ | constant in the differential equation | I-1 |
| $B_2$ | constant in the differential equation | I-1 |
| $C_n$ | constants in expansion of $f(x)$ | I-14 |
| | constants in expansion of $T(x,z)$ | IV-7 |
| D | derivative operator | II-73 |
| E | constant | I-115, II-28, II-136<br>II-28'(ff.III-24) |
| $E_1$ | constant in $y(x)$ | IV-4 |
| $\overline{E}_1$ | constant in $y(x)$ | IV-4 |
| $E_2$ | constant in $Z(z)$ | IV-5 |
| $\overline{E}_2$ | constant in $Z(z)$ | IV-5 |
| F | constant | I-115, II-30, II-136<br>II-30'(ff.III-24) |
| G | constant in subdegree of the solutions | ff.I-40'(ff.I-43)<br>I-48, II-32, III-26 |
| $G^*$ | constant | II-31, III-25 |
| H | constant | I-115, II-94, III-22<br>II-94"(ff.III-73)<br>II-136 |
| $I(o)$ | integrated square of $y_o$ | II-98, III-75 |

| | | |
|---|---|---|
| $I(n)$ | integrated square of $y_n$ | II-101, III-81 |
| $I_1(n)$ | part of $I(n)$ | II-104, III-84 |
| $I_2(n)$ | part of $I(n)$ | II-105, III-85 |
| $J$ | constant | II-26, III-24 |
| | $-v$ | III-76 |
| $K_1$ | constant in the differential equation | I-22 |
| $K_2$ | constant in the differential equation | I-22 |
| $K_1^*$ | constant ratio | II-60 |
| $K_2^*$ | constant ratio | I-50 |
| $L(n)$ | parameter | I-22 |
| $L_1(n)$ | parameter | I-1 |
| $L_2(n)$ | parameter | I-1 |
| $M$ | constant in order of the solutions | III-4 |
| $N$ | constant in order of the solutions | ff.I-35'(ff.I-40)<br>II-12, III-4 |
| $O$ | zero | I-1 |
| $P$ | polynomial in $x$ | II-106, III-86 |
| $Q$ | constant | II-33, III-27 |
| | algebraic function of $x$ | II-122, III-93 |
| | function of $\theta$ | II-113 |
| $Q_1$ | function of $r$ and $\theta$ | II-114 |
| | function of $r, \theta$, and $s$ | II-116 |
| $Q_2$ | function of $r$ and $\theta$ | II-114 |
| | function of $r$, $\theta$, and $s$ | II-116 |

| | | |
|---|---|---|
| R | identifiable function of n | II-73, III-61 |
| | constant | II-75 |
| S | identifiable function of n | II-73, III-61 |
| T | identifiable function of n | II-73, III-61 |
| | function of x and z | IV-1 |
| U | function in integration by parts | ff.II-107, ff.II-113<br>ff.III-87 |
| V | function in integration by parts | ff.II-107, ff.II-113<br>ff.III-87 |
| W | weight function | I-18, II-45, III-37 |
| $Y_n$ | identifiable function of n | II-73, III-61 |
| Z | function of z | IV-2 |
| $Z_n$ | function of z | IV-5 |
| $\bar{Z}_n$ | function of z | IV-5 |

| | | |
|---|---|---|
| B | Beta function | II-76 |
| $\Gamma$ | Gamma function | II-74 |
| $\Pi$ | product | I-30 |
| $\Sigma$ | summation | I-14 |

| | | |
|---|---|---|
| a | an integer | III-10 |
| b | an integer | II-21, III-12 |
| $c^n_{k+\theta j}$ | constants in solutions $y_n$ | I-26 |
| d | differential operator | ff.I-26 |
| e | base of natural logarithms | I-9 |
| f | an arbitrary function of x | I-14, IV-9 |
| g | function of x | II-120, III-90 |
| | function of T,x, and z | IV-1 |
| $g_1$ | function of x and y | IV-3 |
| $g_2$ | function of z and Z | IV-3 |
| g.c.d. | greatest common divisor | II-26, III-21 |
| h | subdegree of descending series solutions | I-48 |
| | order of ascending series solutions | ff.I-56 |
| i | current index | ff.I-14 |
| | constant | ff.I-111, II-131 |
| j | jump | I-27, I-31 |
| $\bar{j}$ | -j | ff.II-78 |
| k | order of descending series solutions | I-26, II-13, III-5 |
| | subdegree of ascending series solutions | ff.I-32 |
| $k_1$ | possibility for k | II-5 |
| $k_2$ | possibility for k | II-6 |
| $k_3$ | possibility for k | II-7 |
| $k_4$ | possibility for k | II-8 |

| | | |
|---|---|---|
| l | unity (on typewriter) | I-1 |
| m | number of solution | ff.I-6 |
| | constant | II-75 |
| n | number of solution | I-1 |
| | current index | I-14 |
| o | zero | I-7 |
| p | coefficient of $y_n''$ in the differential equation | I-2 |
| q | coefficient of $y_n'$ in the differential equation | I-3 |
| r | a power of x in the differential equation | I-1 |
| | current index | ff.II-113 |
| s | a power of x in the differential equation | I-1 |
| | current index | ff.II-114 |
| t | variable of integration | II-75, ff.II-114 |
| $t_n$ | coefficient of $y_n$ in the differential equation | I-4 |
| u | constant ratio | II-9, III-6 |
| v | constant ratio | II-9, III-6 |
| w | integrating factor | I-5 |
| $\bar{w}$ | integrating factor | I-76 |
| x | independent variable | I-1 |
| $x_o$ | a root of wp | I-11 |
| $x_1$ | a root of wp | I-11 |

| | | |
|---|---|---|
| $y_L$ | solutions of the differential equation IV-3 | IV-4 |
| $\bar{y}_L$ | solutions of the differential equation IV-3 | IV-4 |
| $y_n$ | solutions of the differential equation I-1 | I-1 |
| $z$ | new variable of integration | II-115 |
| | independent variable | IV-1 |
| $z_o$ | particular value of $z$ | IV-9 |
| $\theta$ | current index | I-26 |
| | constant | ff.II-75 |
| $\phi$ | function of $x$ | I-13 |
| | current index | I-30 |
| | constant | ff.II-73 |

## DEFINITIONS

| TERM | DEFINED |
|---|---|
| Closed Solution | ff.I-37 |
| Closure Equation | ff.I-39 |
| Derivative Definition | II-95, III-74 |
| Indicial Equation | ff.I-34 |
| Integrated Square | ff.I-13 |
| Jump | ff.I-26 |
| Order of Solutions | ff.I-33 |
| Orthogonality Property | ff.I-11 |
| Orthogonality Relation | ff.I-11 |
| Recursion Product | I-30 |
| Recursion Relation | ff.I-28 |
| Self-Adjoint | ff.I-10 |
| Subdegree of Solutions | ff.I-38 |
| Telescoping | ff.I-109 |
| Weight Function | ff.I-17 |

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# BIOGRAPHY

The writer was born in Toronto, Canada, Nov. 16, 1912. He came to Florida in 1920, and in 1930 graduated from Palm Beach High School as Class Valedictorian, Member National Honor Society, and recipient Sigma Nu Vocational Essay Award. He then entered the University of Florida where he held the competetive State Senatorial Scholarship three years, was a member of the Varsity Glee Club four years, Chancellor Honor Court summer 1937, Member Executive Council summer 1938. He received the B.A. Degree and Reserve Officer Commission June 1934; the B.S. Degree July 1937; the Master's Degree, thesis "Geometry on the Basis of Veblen's Assumptions", August 1940. He has been elected to the Honor Societies ØHΣ and ØKØ Scholastic, and KΔπ Education. His Professional Societies include KØK Education, Charter Membership Florida Academy of Sciences, National Council Teachers of Mathematics, Mathematical Association of America, Reserve Officers Association. From 1934-42 he was a principal and teacher in the Palm Beach County Schools. On active duty 1942-46, he attended Battery Officers and Communications Courses, Ft.Sill; Chinese Language School, University of California, Berkeley; was Communications Officer in India, Indochina, and China, where his book 野戰砲兵通訊課程 was published; decorated with the Bronze Star for Merit and Chinese Cloud and Banner; separated from the service as Major, Signal Corps Reserve. Since 1946 he has been a Mathematics Instructor at the University of Florida.

# APPROVAL

This dissertation was prepared under the direction of the Chairman of the Candidate's Supervisory Committee and has been approved by all Members of the Committee. It was submitted to the Dean of the College of Arts and Sciences, and to the Graduate Council, and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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